Interacting Quantum Field Theory[1]
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1 Interactions and Green functions

In these sections, we discuss perturbation theory for the interacting theory

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4. \tag{1}$$

We will try to find the scattering matrix for this theory. The idea is to consider that $\lambda \ll 1$ and expand in powers of λ. We will follow the argument in Peskin and Schroeder quite closely.

Before we try to find the scattering matrix, we will investigate something a little simpler, the Green functions

$$G(x_1, x_2, \dots, x_N) = \langle\Omega|T\phi(x_1)\phi(x_2)\cdots\phi(x_N)|\Omega\rangle. \tag{2}$$

Here $|\Omega\rangle$ is the vacuum state – the real interacting vacuum state. Following the notation of Peskin and Schroeder we use $|0\rangle$ for the vacuum of the free theory. The T indicates that the field operators are to be time ordered, those with the latest times go to the left. For example, if $t_2 < t_1 < t_3$ then

$$T\phi(x_1)\phi(x_2)\phi(x_3) = \phi(x_3)\phi(x_1)\phi(x_2). \tag{3}$$

To simplify things a little more, let's simply look for the two-point Green function

$$G(x, y) = \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle. \tag{4}$$

and let's assume that $x^0 > y^0$ so that

$$G(x, y) = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle. \tag{5}$$

Once we have solved this problem, it will be easy to generalize to N-point Green functions with any ordering for the times.

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2 The interaction picture

We write

$$H = H_0 + H_{\text{int}} = H_0 + \left(\int d\vec{x}\ \tfrac{1}{4!}\lambda\phi^4 + \Delta\mathcal{E}_0 \right). \tag{6}$$

Here H_0 is the hamiltonian for the free theory and $\Delta\mathcal{E}_0$ is whatever constant we have to add to get the energy of the interacting vacuum to be zero. We write the time dependence of the field as

$$\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt}. \tag{7}$$

Let us also define the *interaction picture field* $\phi_I(t,\vec{x})$ by

$$\phi_I(t,\vec{x}) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}. \tag{8}$$

Then $\phi_I(t,\vec{x})$ is a free field and we can write it in terms of creation and destruction operators as

$$\phi_I(t,\vec{x}) = (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\ \left\{ e^{-ik_\mu x^\mu} a(\vec{k}) + e^{+ik_\mu x^\mu} a^\dagger(\vec{k}) \right\} \tag{9}$$

The interacting field is related to the interaction picture free field by

$$\phi(t,\vec{x}) = U(0,t)\phi_I(t,\vec{x})U(t,0). \tag{10}$$

where

$$U(t,t') = e^{iH_0t}e^{-iH(t-t')}e^{-iH_0t'}. \tag{11}$$

We can also use $U(t,t')$ to relate $|\Omega\rangle$ to $|0\rangle$. We let $|\Omega\rangle$ be the ground state of H, with energy that we have adjusted to be zero, and let $|n\rangle$ be the eigenstates of H with higher energies. Then

$$e^{-iHT}|0\rangle = |\Omega\rangle\langle\Omega|0\rangle + \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle \tag{12}$$

Let T be large with a large negative imaginary part, $T \to \infty \times (1 - i\epsilon)$. In this limit, the $|n\rangle$ terms vanish and we have

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|0\rangle/\langle\Omega|0\rangle = \lim_{T\to\infty(1-i\epsilon)} U(0,-T)|0\rangle/\langle\Omega|0\rangle, \tag{13}$$

and, with a similar argument,

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|e^{-iHT}/\langle 0|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,0)/\langle 0|\Omega\rangle. \tag{14}$$

With these results we can write

$$\begin{aligned} G(x,y) &= \langle\Omega|\phi(x)\phi(y)|\Omega\rangle \\ &= \langle\Omega|U(0,x^0)\phi_I(x)U(x^0,0)U(0,y^0)\phi_I(y)U(y^0,0)|\Omega\rangle \\ &= \langle\Omega|U(0,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,0)|\Omega\rangle \\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{1}{\langle 0|\Omega\rangle\langle\Omega|0\rangle}\langle 0|U(T,0)U(0,x^0)\phi_I(x) \\ &\quad \times U(x^0,y^0)\phi_I(y)U(y^0,0)U(0,-T)|0\rangle \\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|\Omega\rangle\langle\Omega|0\rangle} \end{aligned} \tag{15}$$

We can simplify this a little more by writing the zero-point Green function as

$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,-T)|0\rangle}{\langle 0|\Omega\rangle\langle\Omega|0\rangle} \tag{16}$$

If we divide by 1 written in this way, we get

$$G(x,y) = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}. \tag{17}$$

This is a nice compact expression. To use it, we just have to understand $U(t,t')$.

3 The evolution operator

Well, if we have to understand $U(t,t')$, then we should work with it. From its definition, $U(t,t')$ obeys the differential equation

$$i\frac{d}{dt}U(t,t') = e^{+iH_0t}(H-H_0)e^{-iH_0t}e^{+iH_0t}e^{-iH(t-t')}e^{-iH_0t'} \tag{18}$$

with the boundary condition $U(t',t') = 1$. We can write the differential equation as

$$i\frac{d}{dt}U(t,t') = H_I(t)U(t,t') \tag{19}$$

where

$$H_I(t) = e^{iH_0t}(H-H_0)e^{-iH_0t}. \tag{20}$$

The solution of the differential equation with the boundary condition is

$$U(t,t') = T\exp\left(-i\int_{t'}^{t} d\tau\ H_I(\tau)\right) \tag{21}$$

where the T denotes time ordering of the operators after expanding the exponential. That is

$$\begin{aligned} U(t,t') &= 1 - i\int_{t'}^{t} d\tau\ H_I(\tau) \\ &\quad +(-i)^2\int_{t'}^{t} d\tau_2 \int_{t'}^{\tau_2} d\tau_1\ H_I(\tau_2)\, H_I(\tau_1) \\ &\quad +(-i)^3\int_{t'}^{t} d\tau_3 \int_{t'}^{\tau_3} d\tau_2 \int_{t'}^{\tau_2} d\tau_1\ H_I(\tau_3)\, H_I(\tau_2)\, H_I(\tau_1) \\ &\quad +\cdots. \end{aligned} \tag{22}$$

Note that the Nth order term in the expansion of the exponential has a $1/N!$, which is cancelled because there are $N!$ time orderings of the operators, each giving a contribution of the same form.

Thus, in the result

$$G(x,y) = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}, \tag{23}$$

the denominator is

$$\langle 0|U(T,-T)|0\rangle = \langle 0|T\exp\left(-i\int_{-T}^{T} d\tau\ H_I(\tau)\right)|0\rangle. \tag{24}$$

The numerator is

$$\begin{aligned} &\langle 0|\left\{T\exp\left(-i\int_{x^0}^{T} d\tau\ H_I(\tau)\right)\right\}\phi_I(x)\left\{T\exp\left(-i\int_{y^0}^{x^0} d\tau\ H_I(\tau)\right)\right\} \\ &\times\phi_I(y)\left\{T\exp\left(-i\int_{-T}^{y^0} d\tau\ H_I(\tau)\right)\right\}|0\rangle \end{aligned} \tag{25}$$

Since (we have supposed) $x^0 > y^0$, this is

$$\langle 0|T\exp\left(-i\int_{-T}^{T} d\tau\ H_I(\tau)\right)\phi_I(x)\phi_I(y)|0\rangle. \tag{26}$$

Written in this way, the result works also for $y^0 > x^0$.

4 Result for the Green Function

Thus we arrive at the simple result

$$G(x,y) = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\exp\left(-i\int_{-T}^{T} d\tau\ H_I(\tau)\right)\phi_I(x)\phi_I(y)|0\rangle}{\langle 0|T\exp\left(-i\int_{-T}^{T} d\tau\ H_I(\tau)\right)|0\rangle}. \tag{27}$$

For Green functions with N operators, a similar derivation gives

$$G(x_1,\dots,x_N) = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\exp\left(-i\int_{-T}^{T} d\tau\ H_I(\tau)\right)\phi_I(x_1)\cdots\phi_I(x_N)|0\rangle}{\langle 0|T\exp\left(-i\int_{-T}^{T} d\tau\ H_I(\tau)\right)|0\rangle}. \tag{28}$$

It's easy. Just add some dots!

5 Wick's theorem

If we expand the numerator $\mathcal{N}$ in our result (28) for a Green function we get an expression of the form

$$\mathcal{N}(x_1,\dots,x_N) = \frac{1}{m!}\prod_{j=1}^{m}\left\{\frac{-i\lambda}{4!}\int dz_j\right\}\langle 0|T\phi(x_1)\cdots\phi(x_N)\,\phi^4(z_1)\cdots\phi^4(z_m)|0\rangle. \tag{29}$$

Here each ϕ is an interaction picture field ϕ_I but I leave out the subscript. The $\lambda\phi^4$ factors come from expanding the exponential. The integrand in the exponent contains also a constant term that adjusts the vacuum energy, but this constant term in the exponent creates a constant *factor*, which cancels the corresponding factor in the denominator. Thus I define $\mathcal{N}$ and the corresponding denominator $\mathcal{D}$ to *not* have this factor.

We see that we need to be able to evaluate the (bare) vacuum expectation value of a time ordered product of free fields,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle. \tag{30}$$

Here there are $N+4m$ fields and some of the x s are what are written as z s in Eq. (29). Wick's theorem tells how evaluate Eq. (30).

The first step is to write each ϕ as

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{31}$$

where

$$\begin{aligned}
\phi^+(x) &= (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})} e^{-ik\cdot x} a(k) \\
\phi^-(x) &= (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})} e^{+ik\cdot x} a^\dagger(k).
\end{aligned} \tag{32}$$

Let's start with $n = 2$. Define

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = D_F(x_1 - x_2) \tag{33}$$

It is easy to work out what $D_F(x_1 - x_2)$ is. For $x_1^0 > x_2^0$ we have

$$\begin{aligned}
& D_F(x_1 - x_2) \\
&= \langle 0|\phi(x_1)\phi(x_2)|0\rangle \\
&= \langle 0|\phi^+(x_1)\phi^-(x_2)|0\rangle \\
&= (2\pi)^{-6} \int \frac{d\vec{k}}{2\omega(\vec{k})} e^{-ik\cdot x_1} \int \frac{d\vec{p}}{2\omega(\vec{k})} e^{+ip\cdot x_2} \langle 0|a(k)\, a^\dagger(p)|0\rangle \\
&= (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})} e^{-ik\cdot(x_1 - x_2)} \\
&= (2\pi)^{-4} \int d^4k\; e^{-ik\cdot(x_1-x_2)} \;\frac{2\pi}{2\omega(\vec{k})} \delta(k^0 - \omega(\vec{k})) \\
&= (2\pi)^{-4} \int d^4k\; e^{-ik\cdot(x_1-x_2)} \;\frac{1}{2\omega(\vec{k})} \left\{ \frac{i}{k^0 - \omega(\vec{k}) + i\epsilon} - \frac{i}{k^0 - \omega(\vec{k}) - i\epsilon} \right\} \\
&= (2\pi)^{-4} \int d^4k\; e^{-ik\cdot(x_1-x_2)} \;\frac{1}{2\omega(\vec{k})} \left\{ \frac{i}{k^0 - \omega(\vec{k}) + i\epsilon} - \frac{i}{k^0 + \omega(\vec{k}) - i\epsilon} \right\} \\
&= (2\pi)^{-4} \int d^4k\; e^{-ik\cdot(x_1-x_2)} \;\frac{1}{2\omega(\vec{k})} \left\{ \frac{2i\omega(\vec{k})}{(k^0)^2 - \omega(\vec{k})^2 + i\epsilon} \right\} \\
&= (2\pi)^{-4} \int d^4k\; e^{-ik\cdot(x_1-x_2)} \;\frac{i}{k^2 - m^2 + i\epsilon}
\end{aligned} \tag{34}$$

The most subtle part of this derivation is that with $x_1^0 > x_2^0$ the terms with poles in the upper half k^0 plane give zero contribution and hence can be modified.

Exercise: Show that this works also for $x_2^0 > x_1^0$. Please justify the steps.

We can write the operator product $T\phi(x_1)\phi(x_2)$ itself in a productive way. Define the *normal ordered product* of any number of fields, denoted by $N(\phi(x_1)\cdots\phi(x_N))$, as what you get by decomposing each ϕ into $\phi^+ + \phi^-$ and expanding the result, then in each term moving the ϕ^+ operators to the right of the ϕ^- operators. Within the set of ϕ^+ operators, the ordering doesn't matter since these operators commute. Similarly, within the set of ϕ^- operators, the ordering doesn't matter since these operators commute. The main feature of the normal ordered product is that, since $\langle 0|\phi^- = 0$ and $\phi^+|0\rangle = 0$, we have $\langle 0|N(\phi(x_1)\cdots\phi(x_N))|0\rangle = 0$.

For two field operators we have, if $x_1^0 > x_2^0$,

$$\begin{aligned}
&T\phi(x_1)\phi(x_2) \\
&\quad= \phi^+(x_1)\phi^+(x_2) + \phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \\
&\quad= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_2)\phi^+(x_1) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \\
&\qquad +[\phi^+(x_1),\phi^-(x_2)] \\
&\quad= N(\phi(x_1)\phi(x_2)) + [\phi^+(x_1),\phi^-(x_2)]. \qquad (35)
\end{aligned}$$

Now we recognize that $[\phi^+(x_1),\phi^-(x_2)]$ is a number. If we take the vacuum expectation value of the whole equation, the vacuum expectation value of $N(\phi(x_1)\phi(x_2))$ is zero. Thus the number $[\phi^+(x_1),\phi^-(x_2)]$ is $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$. Thus

$$T\phi(x_1)\phi(x_2) = N(\phi(x_1)\phi(x_2)) + \langle 0|T\phi(x_1)\phi(x_2)|0\rangle, \qquad (36)$$

or

$$T\phi(x_1)\phi(x_2) = N(\phi(x_1)\phi(x_2)) + D_F(x_1 - x_2). \qquad (37)$$

Exercise: Show that this same result holds for $x_2^0 > x_1^0$ also.

In general, Wick's theorem says that

$$T\phi(x_1)\cdots\phi(x_n) = N(\phi(x_1)\cdots\phi(x_n)) + \text{contractions}. \qquad (38)$$

Here "contractions" means that you replace any pair of fields $\{\phi(x_i),\phi(x_j)\}$ in the normal product by $D_F(x_i - x_j)$ and multiply by the normal product of all the rest of the fields. Sum over all possible ways of doing this. Then replace any two pairs of fields by a factor $D_F(x_i - x_j)$ for each pair and multiply by the normal product of all the rest of the fields. Sum over all possible ways of doing this. Next replace all possible combinations of three pairs of fields in the same manner and sum over all the ways to do this. Continuing in this way, the last terms have all fields replaced by $D_F(x_i - x_j)$ factors if n is even, or all but one if n is odd.

The theorem is proved by induction, but it is a easier to explain than to write out the proof. Here are two hints. First, decompose the field into $\phi^+ + \phi^-$ and work on the theorem term by term. Second, use

$$[A, B_1 B_2 B_3 \cdots B_n] = [A, B_1]\, B_2 B_3 \cdots B_n + B_1 [A, B_2] B_3 \cdots B_n + \cdots . \quad (39)$$

The theorem can now be applied to $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$ to give zero of n is odd and, if n is even

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = D_F(x_1 - x_2)\, D_F(x_1 - x_2) \cdots D_F(x_{n-1} - x_n) + \cdots \quad (40)$$

where you have to sum over all ways of pairing the x s.

6 Feynman graphs for Green functions

Recall that the numerator $\mathcal{N}$ in our result for a Green function has the form

$$\mathcal{N}(x_1, \ldots, x_N) = \frac{1}{m!} \prod_{j=1}^{m} \left\{ \frac{-i\lambda}{4!} \int dz_j \right\} \langle 0|T\phi(x_1)\cdots\phi(x_N)\, \phi^4(z_1)\cdots\phi^4(z_m)|0\rangle . \quad (41)$$

For a two-point function at order zero in perturbation theory, we get the bare propagator

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = D_F(x_1 - x_2) = (2\pi)^{-4} \int d^4k\ e^{-ik\cdot(x_1 - x_2)}\ \frac{i}{k^2 - m^2 + i\epsilon} \quad (42)$$

Now we can apply Wick's theorem to Eq. (41) to get rules for expressing the numerator as a sum of integrals.

First, draw a dot, usually refered to as an *external point*, labelled x_i for each $\phi(x_i)$ in Eq. (41). Draw another dot labelled z_i, called a *vertex*, for each $\phi(z_i)^4$ in Eq. (41). Connect the dots with lines representing propagation of a particle from one of the points to another. Each $\phi(x_i)$ external point should have one line attached to it; each $\phi(z_i)^4$ vertex should have four lines attached to it.

- To each line from, say, y_i to y_j associate a propagator $D_F(y_i - y_j)$.
- To each vertex z_i associate a factor $-i\lambda$ and an integration $\int d^4 z_i$.
- To each external point x_i associate a factor 1.
- For the graph as a whole, divide by a symmetry factor (explained below).

The idea of the symmetry factor is to compute each distinct graph only once and multiply by the number of times it occurs according to Wick's theorem, while dividing by the $m!$ and $4!$ factors in Eq. (41). The symmetry factors are straightforward in that a computer program can figure them out, but it is a little awkard to state the rules such a program should use. Here is a set of rules. First, label each vertex, $1,\dots,m$ (and recall the $1/m!$). Furthermore, regard the 4 points that make up each vertex as having distinct label (and recall the $1/4!$). Next, label the propagators $1,\dots,p$ in all possible ways and divide by $1/p!$.

Now, first of all, there is a $1/4!$ associated with each vertex, but there are usually $4!$ different ways to attach 4 differently labelled propagators to the 4 points in the vertex. Thus we can normally just count one of these ways and cancel the $1/4!$. An exception occurs if there is a propagator that connects a vertex to itself. If one propagator connects a certain vertex to itself, there are 4×3 distinct ways to connect the other two propagators, so we have a factor $4 \times 3/4! = 1/2$. If two labelled propagators connect a certain vertex to itself, there are 6 ways to do it, so we have a factor $6/4! = 1/4$. Thus there is a factor $1/2$ for each propagator that connects a vertex to itself.

Second, still counting the vertices as distinct but now counting the four points within a vertex as the same, usually the $p!$ ways to relabel the propagators give different graphs and we get each of these once, so we can just count one of them and cancel the $1/p!$. But if S_P of the relabellings give the *same* graph, the number of times we get this result is smaller by a factor $1/S_P$.

Finally, now counting the four points within a vertex as the same and all of the propagators as the same, usually the $m!$ relabellings of the vertices give distinct graphs, so we can just count one of them and cancel the $1/m!$. But if S_V of the relabellings give the *same* graph, the number of times we get this result is smaller by a factor $1/S_V$.

Thus our symmetry rules are

- A factor 1/2 for each propagator that connects a vertex to itself.
- A factor $1/S_P$ where S_P is the number of permutations of labellings of the propagators that give the same graph when we count the vertices as distinct.
- A factor $1/S_V$ where S_V is the number of permutations of labellings of the vertices that give the same graph (when we count the propagators as identical).

In an actual application, it is a good idea to check the result against Wick's theorem.

7 The denominator

There is also a denominator,

$$\mathcal{D} = \frac{1}{m!} \prod_{j=1}^{m} \left\{ \frac{-i\lambda}{4!} \int dz_j \right\} \langle 0|T\phi^4(z_1)\cdots\phi^4(z_m)|0\rangle. \tag{43}$$

We treat this in the same way. The zero order term is just 1. Then we get all of the Feynman graphs with no external lines, which we can call vacuum graphs.

Now the numerator contains subgraphs with no external lines. That is, some terms in the numerator are connected graphs (all of the vertices are connected to the external points by propagators), while some have the form

$$(\textit{connected graph}) \times (\textit{vacuum graph}) \tag{44}$$

In fact

$$\mathcal{N} = \sum_i (\textit{connected graph})_i \times \left\{ 1 + \sum_j (\textit{vacuum graph})_j \right\}. \tag{45}$$

while the denominator has the form

$$\mathcal{D} = \sum_i \left\{ 1 + \sum_j (\textit{vacuum graph})_j \right\}. \tag{46}$$

Thus the vacuum graph factor in the numerator just cancels the denominator. As a result

$$G(x_1, \ldots, x_N) = \sum_i (\textit{connected graph})_i\,. \tag{47}$$

8 Green functions in momentum space

Let's define Fourier transformed Green functions by

$$\int dx_1 \cdots dx_N \; \exp(-i \sum k_j \cdot x_j)\, G(x_1, \ldots, x_N) = (2\pi)^4\, \delta^{(4)}\Big(\sum k_j\Big)\; \tilde{G}(k_1, \ldots, k_N). \quad (48)$$

We represent $\tilde{G}(k_1, \ldots, k_N)$ by a graph with external lines labelled by the momenta k_j with the momenta coming into the graph.

To evaluate $\tilde{G}(k_1, \ldots, k_N)$, replace each propagator by

$$D_F(x_1 - x_2) = (2\pi)^{-4} \int d^4p \; e^{-ip\cdot(x_1 - x_2)} \; \frac{i}{p^2 - m^2 + i\epsilon} \quad (49)$$

Then at each external point, the p^μ in the propagator gets changed into the external momentum k_j^μ. At each vertex, there is an integral over a z_i that will produce a

$$(2\pi)^4 \delta^{(4)}\left(\sum_i k_i\right). \quad (50)$$

where the k_i^μ are the momenta coming into the vertex. We can use these delta functions to eliminate some of the integrations over the momenta p_i on internal propagator lines. We will be left with one delta function for overall momentum conservation, which we have already factored out of $\tilde{G}(k_1, \ldots, k_N)$. Then we are left with an integration over one propagator momentum for each loop in the diagram. This gives the rules:

- Label the lines by their momenta, using momentum conservation at each vertex.
- For each loop, there will be one momentum that is not constrained by momentum conservation. Supply an integration

$$(2\pi)^{-4} \int d^4p. \quad (51)$$

- To each line associate a propagator

$$\frac{i}{p^2 - m^2 + i\epsilon}. \quad (52)$$

- To each vertex associate a factor $-i\lambda$.
- For the graph as a whole, divide by the symmetry factor.

9 Time flow and the non-relativistic limit

It is of some interest to see what happens if we separate different time orderings in our perturbation theory. Recall what we had for the propagator. If $x_1^0 > x_2^0$, then

$$\begin{aligned} D_F(x_1 - x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \\ &= \langle 0|\phi(x_1)\phi(x_2)|0\rangle \\ &= \langle 0|\phi^+(x_1)\phi^-(x_2)|0\rangle \\ &= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})} e^{-ik\cdot(x_1-x_2)} \\ &= (2\pi)^{-4}\int d^4k\ e^{-ik\cdot(x_1-x_2)}\ \frac{1}{2\omega(\vec{k})}\ \frac{i}{k^0 - \omega(\vec{k}) + i\epsilon}. \end{aligned} \tag{53}$$

If $x_1^0 < x_2^0$, then

$$D_F(x_1 - x_2) = (2\pi)^{-4}\int d^4k\ e^{-ik\cdot(x_1-x_2)}\ \frac{1}{2\omega(\vec{k})}\ \frac{i}{-k^0 - \omega(\vec{k}) + i\epsilon}. \tag{54}$$

(To get this, just change k to $-k$.) Now the expression on the right hand side of Eq. (53) is zero if $x_1^0 < x_2^0$, while the expression on the right hand side of Eq. (54) is zero if $x_1^0 > x_2^0$. Thus if we define

$$D_\pm(x_1 - x_2) = (2\pi)^{-4}\int d^4k\ e^{-ik\cdot(x_1-x_2)}\ \frac{1}{2\omega(\vec{k})}\ \frac{i}{\pm k^0 - \omega(\vec{k}) + i\epsilon} \tag{55}$$

we have

$$D_F(x_1 - x_2) = D_+(x_1 - x_2) + D_-(x_1 - x_2) \tag{56}$$

where $D_+(x_1 - x_2)$ is nonzero for $x_1^0 > x_2^0$ and $D_-(x_1 - x_2)$ is nonzero for $x_1^0 < x_2^0$.

We can break up each propagator in a Feynman graph in this way. A convenient notation for this would be to put an arrow on the line showing which way time flows. Then in momentum space the rule for a propagator is

$$\frac{1}{2\omega(\vec{k})}\ \frac{i}{\pm k^0 - \omega(\vec{k}) + i\epsilon} \tag{57}$$

with the sign in $\pm k^0$ chosen according to the direction of the time flow arrow relative to the momentum flow arrow.

This analysis makes it easy to see what the non-relativistic limit is. Consider a Green function in which some of the lines are "incoming" and some are "outgoing." The incoming lines are associated with early times and for these we choose the momentum signs so that the momentum comes into the graph. The outgoing lines are associated with late times and for these we choose the momentum signs so that the momentum comes out of the graph. Let us suppose that for each incoming and outgoing line we have $p_i^0 \approx m$. Then we can consider the non-relativistic limit in which $m \to \infty$ with $p_i^0 - m$ finite.

Our graph will survive in this limit if, at each vertex, the number of incoming particles equals the number of outgoing particles. This allows us to have also $k_i^0 - m$ for the internal line energies. Then we can approximate

$$\frac{1}{2\omega(\vec{k})}\;\frac{i}{k^0 - \omega(\vec{k}) + i\epsilon} \approx \frac{1}{2m}\;\frac{i}{E_{\rm NR} - \vec{k}^2/(2m) + i\epsilon} \tag{58}$$

where

$$E_{\rm NR} \equiv k^0 - m. \tag{59}$$

This gives a simple non-relativistic theory. We redefine all energies by subtracting the rest energy. The kinetic energy is $\vec{k}^2/(2m)$. For each propagator there is a factor $1/(2m)$ that comes from our normalization of states, which is rather peculiar from a non-relativistic point of view. Associate a factor $q/\sqrt{2m}$ with each external vertex. We can get rid of these by changing normalizations of the field. Also, redefine the coupling by

$$\lambda = (2m)^2 \lambda'. \tag{60}$$

Then all of the $1/(2m)$ factors go away.

Is this a field theory? Yes. For a free field use

$$\psi(t,\vec{x}) = (2\pi)^{-3} \int d\vec{k}\; e^{-iEt + i\vec{k}\cdot\vec{x}} b(\vec{k}), \tag{61}$$

where

$$[b(\vec{k}), b^\dagger(\vec{p})] = (2\pi)^3\, \delta(\vec{k} - \vec{p}). \tag{62}$$

That is, we use just the destruction part of the field and we remove the uninteresting $\exp(-imt)$ part of the time dependence. We also change normalization factors. The commutation relations in position space are

$$\begin{aligned} [\psi(t,\vec{x}), \psi(t,\vec{y})] &= 0 \\ [\psi(t,\vec{x}), \psi^\dagger(t,\vec{y})] &= \delta(\vec{x} - \vec{y}). \end{aligned} \tag{63}$$

This has the interpretation that $\psi(t,\vec{x})$ destroys a particle at position $\vec{x}$ at time t, while $\psi^\dagger(t,\vec{x})$ creates one.

Exercise: Show that Eq. (62) implies $[\psi(t,\vec{x}),\psi^\dagger(t,\vec{y})] = \delta(\vec{x}-\vec{y})$.

The free hamiltonian is

$$H_0 = \int d\vec{x}\ \psi^\dagger(\vec{x}) \frac{-1}{2m} \vec{\nabla}^2\, \psi(\vec{x}) \tag{64}$$

Perhaps a better way to write this is

$$H_0 = \int d\vec{x}\ \frac{1}{2m} \vec{\nabla}\psi^\dagger(t,\vec{x}) \cdot \vec{\nabla}\psi(t,\vec{x}) \tag{65}$$

We add interactions by defining

$$H_{\rm int} = \int d\vec{x}\ \frac{\lambda'}{4} (\psi^\dagger(t,\vec{x}))^2 (\psi(t,\vec{x}))^2. \tag{66}$$

One can check that this gives the Feynman rules that we have derived by taking the $m\to\infty$ limit of the relativistic theory.

Exercise: What equation of motion for $\psi(\vec{x},t)$ follows from the hamiltonian $H_0 + H_{\rm int}$ in Eqs. (65) and (66)?

We can extend this idea to particles that interact by means of a potential. We take

$$H_{\rm int} = \int d\vec{x} \int d\vec{y}\ \tfrac{1}{2} \psi^\dagger(t,\vec{x})\psi^\dagger(t,\vec{y})\, \psi(t,\vec{x})\psi(t,\vec{y})\ \mathcal{V}(\vec{x}-\vec{y}). \tag{67}$$

Then the interaction (66) is a special case of this with a delta function potential.

Exercise: Evaluate $H_{\rm int}|\vec{x},\vec{y}\rangle$, where $H_{\rm int}$ is evaluated at $t=0$ and the state $|\vec{x},\vec{y}\rangle$ is what you get by applying $\psi^\dagger(0,\vec{x})\psi^\dagger(0,\vec{y})$ to the vacuum state.

Suppose that the Fourier transform of $\mathcal{V}$ is

$$\tilde{\mathcal{V}}(\vec{q}) = \int d\vec{x}\ e^{-i\vec{q}\cdot\vec{x}}\, \mathcal{V}(\vec{x}). \tag{68}$$

Then the Feynman rules for Green functions includes a new rule. We can represent the action of the potential by, say, a dashed line that couples to an incoming and an outgoing particle line at each end. The line carries momentum $\vec{q}$ determined by momentum conservation. We write

- For each interaction, a factor $-i\tilde{\mathcal{V}}(\vec{q})$.

What if we take one step backward and look for an action for our theory. This works. The action is

$$
\begin{aligned}
S[\psi^\dagger,\psi] &= \int dt\, d\vec{x}\, \Big\{\psi^\dagger(t,\vec{x})[\tfrac{i}{2}\partial_t\psi(t,\vec{x})] - [\tfrac{i}{2}\partial_t\psi^\dagger(t,\vec{x})]\psi(t,\vec{x}) \\
&\qquad -\frac{1}{2m}\vec{\nabla}\psi^\dagger(t,\vec{x})\cdot\vec{\nabla}\psi(t,\vec{x})\Big\} \\
&\quad -\int dt\, d\vec{x}\, d\vec{y}\ \tfrac{1}{2}\psi^\dagger(t,\vec{x})\psi^\dagger(t,\vec{y})\,\psi(t,\vec{x})\psi(t,\vec{y})\ \mathcal{V}(\vec{x}-\vec{y}). \qquad (69)
\end{aligned}
$$

Exercise: With this action, evaluate the equation of motion

$$
\frac{\delta S[\psi^\dagger,\psi]}{\delta\psi^\dagger(t,\vec{x})} = 0. \qquad (70)
$$

Having the action is useful for deriving the conservation laws and the corresponding conserved quantities from Nöther's theorem. It is also useful because the action comes into the path integral formulation of the quantum field theory (to be discussed later in the course). But the relation of the commutation relations to Poisson brackets is different than the previous cases we have discussed. That is because the formulation we used for the relativistic theory is really designed for theories with equations of motion that are second order in $\partial/\partial t$. The Schrödinger equation is first order in time derivatives.

Exercise: The action (69) is invariant under translations in space and time. What are the corresponding conserved quantities? You will have to derive a new version of Nöther's theorem to cover this case.

---

10 Charged scalar field

So far we have talked about a field that destroys and creates particles that are their own antiparticles (like photons). With just a few small modifications, we can have a field that destroys and creates particles that are *not* their own antiparticles like, say, ^{4}He nuclei. Since we are making modifications to a theory that we already understand, we can be brief.

Consider the lagrangian

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi - \frac{\lambda}{4}(\phi^\dagger \phi)^2. \tag{71}$$

Here $\phi^\dagger$ is the adjoint of ϕ, but in doing the algebra it is convenient to regard $\phi^\dagger$ and ϕ as independent fields. If we want we can write $\phi = \phi_1 + i\phi_2$, where ϕ_1 and ϕ_2 are real (*ie.* self-adjoint) fields. Then we can vary ϕ_1 and ϕ_2 independently.

The equations of motion are

$$0 = \frac{\delta S[\phi^\dagger, \phi]}{\delta \phi^\dagger(x)} = (-\partial_\mu \partial^\mu - m^2)\phi(x) - \frac{\lambda}{2}(\phi^\dagger(x)\phi(x))\phi(x) \tag{72}$$

and

$$0 = \frac{\delta S[\phi^\dagger, \phi]}{\delta \phi(x)} = (-\partial_\mu \partial^\mu - m^2)\phi^\dagger(x) - \frac{\lambda}{2}(\phi^\dagger(x)\phi(x))\phi^\dagger(x) \tag{73}$$

which is just the adjoint of the first equation.

To put this in the hamiltonian formulation, we define

$$\pi(\vec{x}) = \frac{\delta L[\dot{\phi}^\dagger, \phi^\dagger, \dot{\phi}, \phi]}{\delta \dot{\phi}(\vec{x})} = \dot{\phi}^\dagger(\vec{x}) \tag{74}$$

and

$$\pi^\dagger(\vec{x}) = \frac{\delta L[\dot{\phi}^\dagger, \phi^\dagger, \dot{\phi}, \phi]}{\delta \dot{\phi}^\dagger(\vec{x})} = \dot{\phi}(\vec{x}). \tag{75}$$

Then the hamiltonian density is

$$\mathcal{H} = \pi\dot{\phi} + \pi^\dagger \dot{\phi}^\dagger - \mathcal{L} = \pi^\dagger \pi + (\vec{\nabla}\phi^\dagger)\cdot(\vec{\nabla}\phi) + m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2. \tag{76}$$

The commutation relations are

$$\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= i\delta(\vec{x} - \vec{y}) \\ [\phi^\dagger(\vec{x}), \pi^\dagger(\vec{y})] &= i\delta(\vec{x} - \vec{y}). \end{aligned} \tag{77}$$

One can check that with these commutation relations, the hamiltonian generates the right equations of motion.

Now for a free field ($\lambda = 0$) we can solve the theory exactly and write

$$\phi(t,\vec{x}) = (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})} \left\{ e^{-ik_\mu x^\mu} a(\vec{k}) + e^{+ik_\mu x^\mu} b^\dagger(\vec{k}) \right\} \tag{78}$$

Here $a(\vec{k})$ destroys a particle and $b^\dagger(\vec{k})$ creates something different – an antiparticle. The commutation relations are

$$\begin{aligned} [a(\vec{k}), a^\dagger(\vec{p})] &= (2\pi)^3\, 2\omega(\vec{k})\, \delta(\vec{k} - \vec{p}) \\ [b(\vec{k}), b^\dagger(\vec{p})] &= (2\pi)^3\, 2\omega(\vec{k})\, \delta(\vec{k} - \vec{p}). \end{aligned} \tag{79}$$

These are equivalent to the ϕ, π commutation relations given above.

Now to do perturbation theory, we imitate the previous treatment, using

$$\begin{aligned} \langle 0|T\phi(x_1)\phi^\dagger(x_2)|0\rangle &= D_F(x_1 - x_2) \\ &= (2\pi)^{-4} \int d^4k\; e^{-ik\cdot(x_1 - x_2)}\; \frac{i}{k^2 - m^2 + i\epsilon} \end{aligned} \tag{80}$$

We represent this in graphs as a line with an arrow going from x_2 to x_1. The arrow reminds us that we create a particle at x_2 and annihilate it at x_1. Or else we create an antiparticle at x_1 and annihilate it at x_2. Thus the arrow represents the direction of particle flow. Note the difference in convention with our (rather non-conventional) use of an arrow to represent the direction of time flow in previous sections.

As before, we can easily derive the Feynman rules for Green functions for this theory. We draw all connected graphs for the desired Green function. Now the propagators have arrows representing the direction of particle flow. At each vertex, there are two incoming arrows and two outgoing arrows.

- Label the lines by their momenta, using momentum conservation at each vertex.

- For each loop, there will be one momentum that is not constrained by momentum conservation. Supply an integration

$$(2\pi)^{-4} \int d^4p. \tag{81}$$

- To each line associate a propagator

$$\frac{i}{p^2 - m^2 + i\epsilon}. \tag{82}$$

- To each vertex associate a factor $-i\lambda$.
- For the graph as a whole, divide by the symmetry factor.

Exercise: What are the Feynman rules for Green functions for a theory with one real field ϕ and one complex field ψ with the lagrangian density

$$\mathcal{L} = (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - m^2 \psi^\dagger \psi + \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - g\phi\, \psi^\dagger \psi \tag{83}$$

---

11 Scattering

Let's define a Green function for our standard ϕ^4 theory with some of the momenta, denoted k_i, going into the graph and some, denoted q_i, coming out

$$\begin{aligned}\int dy_1 \cdots dy_M\, dx_1 \cdots dx_N\; \exp(i \textstyle\sum q_j \cdot y_j - i \sum k_j \cdot x_j)\\ \times \langle \Omega | T\phi(y_1) \cdots \phi(y_M)\phi(x_1) \cdots , \phi(x_N) | \Omega \rangle\\ = (2\pi)^4\, \delta^{(4)}\Big(\textstyle\sum k_j - \sum q_j\Big)\; \tilde{G}(q_1 \ldots , q_M; k_1, \ldots , k_N).\end{aligned} \tag{84}$$

We would like to use $\tilde{G}(q_1 \ldots , q_M; k_1, \ldots , k_N)$ to describe the scattering matrix

$$S_{FI} = {}_{\rm out}\langle q_1 \ldots , q_M | k_1, \ldots , k_N \rangle_{\rm in} \tag{85}$$

(Please refer to the notes on the S-matrix from last quarter for the definitions.)

The relation is often called the LSZ reduction formula (Lehmann, Symanzik & Zimmermann). To start with, we note that with our normalizations in a free field theory a field can annihilate a particle to give the vacuum state with amplitude 1:

$$\langle \Omega | \phi(0) | k \rangle = 1 \qquad \text{(free theory)}. \tag{86}$$

We suppose that we are near enough to this situation that a field can annihilate a particle to give the vacuum state, just with some different amplitude, which we call $\sqrt{Z}$:

$$\langle \Omega | \phi(0) | k \rangle = \sqrt{Z}. \tag{87}$$

(We choose the phase of the states so that $\sqrt{Z}$ is real and positive.) With this definition, the LSZ formula is

$$\begin{aligned}
&(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\,\tilde{G}(q_1\ldots,q_M;k_1,\ldots,k_N) \\
&\sim\ \prod_{j=1}^{M}\frac{i\sqrt{Z}}{q_i^2-M^2+i\epsilon}\ {}_{\rm out}\langle q_1\ldots,q_M|k_1,\ldots,k_N\rangle_{\rm in}\ \prod_{j=1}^{N}\frac{i\sqrt{Z}}{k_i^2-M^2+i\epsilon} \qquad (88)
\end{aligned}$$

Here M is the physical mass of the particles (which may not be the same as the mass m in the lagrangian). Here the "$\sim$" means asymptotic equality as one approaches the poles at $q_i^2 = M^2$, $k_i^2 = M^2$. That is, the equation asserts that both sides of the equation have poles at $q_i^2 = M^2$, $k_i^2 = M^2$ and the residues of these poles match. Thus

$$\begin{aligned}
{}_{\rm out}\langle q_1\ldots,q_M|k_1,\ldots,k_N\rangle_{\rm in} \;=\; &(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\times \lim_{\substack{q_i^2\to M^2\\ k_i^2\to M^2}}\Big\{ \\
&\times\prod_{j=1}^{M}\frac{q_i^2-M^2+i\epsilon}{i\sqrt{Z}}\ \prod_{j=1}^{N}\frac{k_i^2-M^2+i\epsilon}{i\sqrt{Z}} \\
&\times\tilde{G}(q_1\ldots,q_M;k_1,\ldots,k_N)\Big\} \qquad (89)
\end{aligned}$$

To see why this is at least plausible, let's look first at the two point function.

$$\tilde{G}(k) = \int dx\,\exp(-ik\cdot x)\langle\Omega|T\phi(0)\phi(x)|\Omega\rangle. \qquad (90)$$

Exercise: Show that this is the same as

$$(2\pi)^4\delta^4(q-k)\,\tilde{G}(k) = \int dx\,dy\,\exp(-ik\cdot x + iq\cdot y)\langle\Omega|T\phi(y)\phi(x)|\Omega\rangle. \qquad (91)$$

You can use

$$\exp(iP\cdot x)\phi(y)\exp(-iP\cdot x) = \phi(x+y) \qquad (92)$$

where P^μ is the energy-momentum operator. This is equavalent to

$$i[P^\mu,\phi(y)] = \partial^\mu\phi(y) \qquad (93)$$

which are the standard commutation relations between P^μ and the fields.

We will look for poles in $\tilde{G}(k)$ at $k^2 = M^2$ with $k^0 > 0$. Such a pole can come only from the integration region $x^0 \to -\infty$, as will become apparent from the construction below. Accepting this for the moment, we write

$$\tilde{G}(k) \sim \int d\vec{x} \int_{-\infty}^{-T} dx^0 \exp(-ik \cdot x) \langle \Omega | \phi(0) \phi(x) | \Omega \rangle. \tag{94}$$

Also if we intsert intermediate states between the $\phi(0)$ and the $\phi(x)$,the pole can come only from single particle states, as will become apparent from the construction below. Accepting this for the moment, we write

$$\tilde{G}(k) \sim \int_{-\infty}^{-T} dx^0 \int d\vec{x}\ \exp(-ik \cdot x)\, (2\pi)^{-3} \int \frac{d\vec{p}}{2\omega(\vec{p})} \langle \Omega | \phi(0) | p \rangle \langle p | \phi(x) | \Omega \rangle. \tag{95}$$

Translation invariance, Eq. (92), gives

$$\langle p | \phi(x) | \Omega \rangle = \exp(ip \cdot x) \langle p | \phi(0) | \Omega \rangle \tag{96}$$

Then, using the definition of Z, we have

$$\tilde{G}(k) \sim Z \int_{-\infty}^{-T} dx^0 \int d\vec{x}\ (2\pi)^{-3} \int \frac{d\vec{p}}{2\omega(\vec{p})} \exp(-i(k-p) \cdot x)\,. \tag{97}$$

Performing the $\vec{x}$ integral and using $p^0 = \omega(\vec{p})$ we have

$$\tilde{G}(k) \sim Z \int_{-\infty}^{-T} dx^0\ \frac{1}{2\omega(\vec{k})} \exp(-i(k^0 - \omega(\vec{k}))x^0)\,. \tag{98}$$

This is

$$\tilde{G}(k) \sim \frac{Z}{2\omega(\vec{k})} \frac{i}{k^0 - \omega(\vec{k}) + i\epsilon} \exp(iT(k^0 - \omega(\vec{k}))). \tag{99}$$

At the pole, we can relplace the exponential by 1 and we can replace $2\omega(\vec{p})$ by $k^0 + \omega(\vec{k})$. Then

$$\tilde{G}(k) \sim Z \frac{i}{(k^0)^2 - \omega(\vec{k})^2 + i\epsilon}. \tag{100}$$

That is

$$\tilde{G}(k) \sim \frac{iZ}{k^2 - M^2 + i\epsilon}. \tag{101}$$

Now that we see how this works, we can go back and notice that the part of the integration over x^0 over any finite time interval, say from T_1 to T_2 does not give a pole. Repeating essentially the same argument, the part from some finite time to $+\infty$ gives a pole of the same form along the negative energy hyperbola $k^2 = M^2$ with $k^0 < 0$. Repeating the argument, we get more singularities in k^2 starting at the lowest mass squared for a two particle state, $k^2 = (2M)^2$. Since two particle states can have masses M_2 that are anything down to $2M$, the singularity at $k^2 = (2M)^2$ is not isolated. (It is a branch point.) Thus Eq. (101) gives the structure of $\tilde{G}(k)$ as one approaches the one particle pole only.

Eq. (101) is what one uses to calculate Z in perturbation theory. You calculate $\tilde{G}(k)$ at some order of perturbation theory and extract the residue of the one particle pole. This pole will generally have moved from $k^2 = m^2$ to somewhere else, $k^2 = M^2$, so one calculates M^2 too.

Now we would like to make the LSZ formula at least plausible, following the argument in Peskin and Schroeder Chapter 7.2. We start with

$$
\begin{aligned}
(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \ &\tilde{G}(q_1\dots,q_M;k_1,\dots,k_N) \\
&= \int dy_1\cdots dy_M\, dx_1\cdots dx_N\, \exp(i\sum q_j\cdot y_j - i\sum k_j\cdot x_j) \\
&\times\langle\Omega|T\phi(y_1)\cdots\phi(y_M)\phi(x_1)\cdots,\phi(x_N)|\Omega\rangle
\end{aligned}
\tag{102}
$$

We are going to look for poles at the $k_j^2 = M^2$ and $q_j^2 = M^2$ with positive energies, $k_j^0 > 0$, $q_j^0 > 0$. As we learned with the two point function, these poles come from the part of the x_j^0 integration with $x_j^0 \to -\infty$ and the part of the y_j^0 integration with $y_j^0 \to +\infty$. Thus we can write

$$
\begin{aligned}
(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \ &\tilde{G}(q_1\dots,q_M;k_1,\dots,k_N) \\
\sim\ \ &\int dy_1\cdots dy_M\,\prod_j \theta(y_j^0 > T)\exp(i\sum q_j\cdot y_j) \\
&\times\int dx_1\cdots dx_N\,\prod_j \theta(x_j^0 < -T)\exp(-i\sum k_j\cdot x_j) \\
&\times\langle\Omega|T\{\phi(y_1)\cdots\phi(y_M)\}T\{\phi(x_1)\cdots,\phi(x_N)\}|\Omega\rangle
\end{aligned}
\tag{103}
$$

Now we insert a sum over a complete set of out state just to the right of the $\phi(y)$ factors and a sum over a complete set of in state just to the left of the $\phi(x)$ factors:

$$
(2\pi)^4\,\delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \ \tilde{G}(q_1\dots,q_M;k_1,\dots,k_N)
$$

$$
\begin{aligned}
\sim \quad & \int dy_1 \cdots dy_M \prod_j \theta(y_j^0 > T) \exp(i \sum q_j \cdot y_j) \\
& \times \sum_K \frac{1}{K!} \prod_{i=1}^{K} (2\pi)^{-3} \int \frac{d\vec{p}_i}{2\omega(\vec{p}_i)} \\
& \times \langle \Omega | T\{\phi(y_1) \cdots \phi(y_M)\} | p_1, \ldots, p_K \rangle_{\rm out} \\
& \times \sum_L \frac{1}{L!} \prod_{j=1}^{L} (2\pi)^{-3} \int \frac{d\vec{l}_j}{2\omega(\vec{l}_j)} \\
& \times \int dx_1 \cdots dx_N \prod_j \theta(x_j^0 < -T) \exp(-i \sum k_j \cdot x_j) \\
& \times {}_{\rm in}\langle l_1, \ldots l_L | T\{\phi(x_1) \cdots, \phi(x_N)\} | \Omega \rangle \\
& \times {}_{\rm out}\langle p_1, \ldots p_K | l_1, \ldots l_L \rangle_{\rm in}
\end{aligned} \tag{104}
$$

Now lets look at the matrix element of the $\phi(y)$ fields. As we shall see, we can get the requisite poles at $q_i^2 = M^2$ out of this. The way it can come about is if each field operator destroys one particle. Then we should have $K = N$ and $L = M$. The essential approximation for the final state particles is

$$
\begin{aligned}
\langle \Omega | T\{\phi(y_1) \cdots \phi(y_M)\} | p_1, \ldots, p_K \rangle_{\rm out} \quad \rightarrow \quad & \langle \Omega | \phi(y_1) | p_1 \rangle_{\rm out} \cdots \langle \Omega | \phi(y_M) | p_M \rangle_{\rm out} \\
& + \cdots
\end{aligned} \tag{105}
$$

where the omitted terms are of the same type, but with the indices of the ys matched up in a different order with the indices of the ps. There are $M!$ such terms, and each will give the same contribution to the final answer, so we will just keep the one term displayed above and cancel the factor $1/K!$ in the formula.

The idea of this replacement is that the y_i^0 are very large and in the out state after a long time the particles are all a long way from each other, so each field operator annihilates one particle independently of what is happening with the other particles. Of course, to really make this sensible, we should put the particles into wave packet states, so that their wave functions have very little overlap in space after a long time. We could also integrate the field operators against wave packet wave functions instead of plane waves. However, I will skip further explorations along these lines.

For the in states we make the same kind of replacement

$$
\begin{aligned}
{}_{\rm in}\langle p_1, \ldots, p_N | T\{\phi(x_1) \cdots \phi(x_N)\} | \Omega \rangle \quad \rightarrow \quad & {}_{\rm in}\langle p_1 | \phi(x_1) | \Omega \rangle \cdots {}_{\rm in}\langle p_N | \phi(x_N) | \Omega \rangle \\
& + \cdots
\end{aligned} \tag{106}
$$

Then we have

$$
\begin{aligned}
(2\pi)^4\, \delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \ \tilde G(q_1 \dots, q_M; k_1, \dots, k_N) \\
\sim\ \prod_j \int dy_j\, \theta(y_j^0 > T)\exp(iq_j \cdot y_j)\,(2\pi)^{-3}\int \frac{d\vec p_j}{2\omega(\vec p_j)}\langle\Omega|\phi(y_j)|p_j\rangle_{\rm out} \\
\times \prod_i \int dx_i\, \theta(x_i^0 < -T)\exp(-ik_i \cdot x_i)\,(2\pi)^{-3}\int \frac{d\vec l_i}{2\omega(\vec l_i)}\ {}_{\rm in}\langle p_i|\phi(x_i)|\Omega\rangle \\
\times\ {}_{\rm out}\langle p_1, \dots p_M | l_1, \dots l_N\rangle_{\rm in}
\end{aligned}
\tag{107}
$$

Performing the integrals gives

$$
\begin{aligned}
(2\pi)^4\, \delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \ \tilde G(q_1 \dots, q_M; k_1, \dots, k_N) \\
\sim\ \prod_j \frac{i\sqrt{Z}}{q_j^2 - M^2 + i\epsilon}\ \prod_i \frac{i\sqrt{Z}}{k_i^2 - M^2 + i\epsilon} \\
\times\ {}_{\rm out}\langle q_1, \dots q_M | k_1, \dots k_N\rangle_{\rm in},
\end{aligned}
\tag{108}
$$

which is the LSZ formula.

We should note that the Green function and the S-matix both contain terms in which one or more particles go from the initial state to the final state without interacting at all. This derivation does not apply to those "no scattering" terms, so let's understand that we have subtracted these terms from both sides of the equation.

Note that the possibility of having non-trivial factors $\sqrt{Z}$ arises because a particle can interact with itself even after it is far from any other particles. In standard non-relativistic physics this doesn't happen, so $\sqrt{Z} = 1$.

12 The S-matrix and amputated Green functions

We can write the Green function discussed in the previous section as

$$
\tilde G(q_1 \dots, q_M; k_1, \dots, k_N) = \prod_j \tilde G_2(q_j)\ \prod_i \tilde G_2(k_i)\ \tilde\Gamma(q_1 \dots, q_M; k_1, \dots, k_N)
\tag{109}
$$

Here $\tilde G_2$ is the two particle Green function, called the "full propagator," and $\tilde\Gamma$ is the "amputated Green function" and is defined by this eequation.

Graphically, we just omit the extermal legs of the diagram, including any interactions on the external legs. One says that $\tilde{\Gamma}$ is *one particle irreducible.*

We have

$$\tilde{G}_2(k_i) \sim \frac{iZ}{k_i^2 - M^2 + i\epsilon} \tag{110}$$

at the single particle pole. Since we calculate the S-matrix by factoring out an

$$\frac{i\sqrt{Z}}{k_i^2 - M^2 + i\epsilon} \tag{111}$$

for each external particle, we can restate the prescription as telling us to factor out the full propagators, leaving the amputated Green function, and *multiply* by $\sqrt{Z}$:

$$\begin{aligned} {}_{\text{out}}\langle q_1 \ldots, q_M | k_1, \ldots, k_N \rangle_{\text{in}} &= (2\pi)^4\, \delta^{(4)}\Big(\sum k_j - \sum q_j\Big)\ \left(\sqrt{Z}\right)^{N+M} \\ &\quad \times \lim_{\substack{q_i^2 \to M^2 \\ k_i^2 \to M^2}} \tilde{\Gamma}(q_1 \ldots, q_M; k_1, \ldots, k_N). \end{aligned} \tag{112}$$