Free Scalar Field Theory[1]
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Physics 665, Quantum Field Theory
January 2001

1 The theory

We consider the quantum theory that arises from the lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2}m^2\phi^2. \tag{1}$$

The equation of motion is

$$[\partial_\mu \partial^\mu + m^2]\phi = 0. \tag{2}$$

The canonical momentum conjugate to φ is

$$\pi = \frac{\partial\phi}{\partial t}. \tag{3}$$

The hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi) + \tfrac{1}{2}m^2\varphi^2. \tag{4}$$

The fields φ and π become operators. Since the classical fields are real, we take the quantum fields to be hermitian operators. The commutation relations are

$$\begin{aligned} [\varphi(\vec{x}), \varphi(\vec{y})] &= 0 \\ [\pi(\vec{x}), \pi(\vec{y})] &= 0 \\ [\varphi(\vec{x}), \pi(\vec{y})] &= i\delta(\vec{x} - \vec{y}). \end{aligned} \tag{5}$$

We address the question, "What physical system does this describe."

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2 Solving the equations of motion

Let us define Fourier transform variables

$$\begin{aligned}\tilde{\varphi}(\vec{k}) &= \int d\vec{x}\; e^{-i\vec{k}\cdot\vec{x}}\varphi(\vec{x}) \\ \tilde{\pi}(\vec{k}) &= \int d\vec{x}\; e^{-i\vec{k}\cdot\vec{x}}\pi(\vec{x}).\end{aligned} \tag{6}$$

The fields $\tilde{\varphi}$ and $\tilde{\pi}$ are not hermitian, but obey

$$\begin{aligned}\tilde{\varphi}(\vec{k})^\dagger &= \tilde{\varphi}(-\vec{k}) \\ \tilde{\pi}(\vec{k})^\dagger &= \tilde{\pi}(-\vec{k}).\end{aligned} \tag{7}$$

We also define $\omega(\vec{k})$ as the energy of a free particle with momentum $\vec{k}$:

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}. \tag{8}$$

We use $\tilde{\varphi}$ and $\tilde{\pi}$ just as an intermediate step along the way to defining

$$\begin{aligned}a(\vec{k}) &= \omega(\vec{k})\,\tilde{\varphi}(\vec{k}) + i\tilde{\pi}(\vec{k}) \\ a^\dagger(\vec{k}) &= \omega(\vec{k})\,\tilde{\varphi}(-\vec{k}) - i\tilde{\pi}(-\vec{k}).\end{aligned} \tag{9}$$

Then

$$\begin{aligned}\varphi(\vec{x}) &= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;\left\{e^{i\vec{k}\cdot\vec{x}}a(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}}a^\dagger(\vec{k})\right\} \\ \pi(\vec{x}) &= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;[-i\omega(\vec{k})]\left\{e^{i\vec{k}\cdot\vec{x}}a(\vec{k}) - e^{-i\vec{k}\cdot\vec{x}}a^\dagger(\vec{k})\right\}.\end{aligned} \tag{10}$$

Exercise. Verify Eq. (10).

The dynamical variables φ and π depend on the time, although I have not indicated this dependence explicitly. Thus the derived dynamical variables a also depend on the time. From the equations of motion for φ and π we find

$$\frac{\partial}{\partial t}\,a(t;\vec{k}) = -i\omega(\vec{k})\,a(t;\vec{k}). \tag{11}$$

Exercise. Verify Eq. (11).

Eq. (11) was the reason for defining the a variables. We have achieved an ordinary differential equation instead of a partial differential equation as the equation of motion. We can solve this trivially:

$$a(t;\vec{k}) = e^{-i\omega(\vec{k})t} a(0;\vec{k}). \tag{12}$$

Then Eq. (10) becomes

$$\varphi(t,\vec{x}) = (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})}\ \left\{e^{-ik_\mu x^\mu} a(\vec{k}) + e^{+ik_\mu x^\mu} a^\dagger(\vec{k})\right\} \tag{13}$$

with $k^0 \equiv \omega(\vec{k})$. The corresponding equation for π can be obtained using this result and $\pi = \partial\varphi/\partial t$.

3 Commutation relations

With a little effort, one finds

$$\begin{aligned} [a(\vec{k}), a(\vec{p})] &= 0 \\ [a(\vec{k}), a^\dagger(\vec{p})] &= (2\pi)^3\, 2\omega(\vec{k})\, \delta(\vec{k} - \vec{p}). \end{aligned} \tag{14}$$

Exercise. Verify Eq. (14).

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4 Energy and momentum operators

If we express the hamiltonian as a function of a and $a^\dagger$, we get

$$H = (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})}\ \omega(\vec{k})\, \tfrac{1}{2}[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})]. \tag{15}$$

Similarly, for the momentum operator we get

$$P^j = (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})}\ k^j\, \tfrac{1}{2}[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})]. \tag{16}$$

Exercise. Prove Eqs. (15) and (16).

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5 How a changes energy and momentum

Using the canonical commutation relations and our equations for H and P^j we derive

$$\begin{aligned} [H, a(\vec{k})] &= -\omega(\vec{k})\, a(\vec{k}) \\ [P^j, a(\vec{k})] &= -k^j\, a(\vec{k}). \end{aligned} \tag{17}$$

Taking the adjoint of these, we have

$$\begin{aligned} [H, a(\vec{k})^\dagger] &= +\omega(\vec{k})\, a^\dagger(\vec{k}) \\ [P^j, a(\vec{k})^\dagger] &= +k^j\, a^\dagger(\vec{k}). \end{aligned} \tag{18}$$

Exercise. Verify Eq. (17).

These equations are very instructive. Suppose that we have a state that is an eigenstate of energy and momentum with eigenvalues $p^m u$,

$$P^\mu|\psi\rangle = p^\mu|\psi\rangle. \tag{19}$$

Act on this with $a(\vec{k})^\dagger$. What is the energy and momentum of the resulting state? To find out, use the commutation relations, denoting $k^0 = \omega(\vec{k})$,

$$\begin{aligned} P^\mu\, a(\vec{k})^\dagger|\psi\rangle &= a(\vec{k})^\dagger\, P^\mu|\psi\rangle + k^\mu a(\vec{k})^\dagger|\psi\rangle \\ &= a(\vec{k})^\dagger\, p^\mu|\psi\rangle + k^\mu a(\vec{k})^\dagger|\psi\rangle \\ &= (p^\mu + k^\mu)\, a(\vec{k})^\dagger|\psi\rangle. \end{aligned} \tag{20}$$

Similarly, if we act on $|\psi\rangle$ with a we have

$$P^\mu\, a(\vec{k})|\psi\rangle = (p^\mu - k^\mu)\, a(\vec{k})|\psi\rangle. \tag{21}$$

Thus $a(\vec{k})^\dagger$ raises the energy and momentum of the state by k^μ while $a(\vec{k})$ lowers the energy and momentum of the state by k^μ. The natural interpretation is that $a(\vec{k})^\dagger$ creates a particle with energy-momentum k^μ while $a(\vec{k})$ destroys a particle with energy-momentum k^μ.

We just need to postulate the existence of a vacuum state, $|0\rangle$, with $P^\mu|0\rangle = 0$. Then a one particle state with a fixed momentum is $|k\rangle = a^\dagger(\vec{k})|0\rangle$. A two particle state is $|k, p\rangle = a^\dagger(\vec{k})a^\dagger(\vec{p})|0\rangle$, *etc.*

6 Fock space

Let's look in more detail at the space of states. We first suppose that there is a vacuum state $|0\rangle$ with the properties

$$\begin{aligned} \langle 0|0\rangle &= 1 \\ P^\mu|0\rangle &= 0 \\ a(k)|0\rangle &= 0. \end{aligned} \tag{22}$$

(In this section, $a(k) = a(t,\vec{k})$ at $t = 0$. I use a covariant notation in which k is a four-vector with $k^0 = \omega(\vec{k})$.) We have to add a constant to P^μ to make the 4-momentum of the vacuum zero; we discuss this later.

The one particle states are

$$|k\rangle = a^\dagger(k)\,|0\rangle. \tag{23}$$

Our previous analysis then gives

$$P^\mu|k\rangle = k^\mu\,|k\rangle. \tag{24}$$

With a little work, we can find how the particles transform under Lorentz transformations. We need the generators of Lorentz transformations, $J^{\alpha\beta}$, for this. Let's leave out the details for now. We get

$$U(\Lambda)|k\rangle = |\Lambda k\rangle. \tag{25}$$

This is the transformation law for spin zero particles. Thus our particles have spin zero.

The normalization is

$$\begin{aligned} \langle p|k\rangle &= \langle 0|a(p)a^\dagger(k)|0\rangle \\ &= \langle 0|[a(p),a^\dagger(k)]|0\rangle + \langle 0|a^\dagger(k)\,a(p)|0\rangle \\ &= (2\pi)^3 2\omega(\vec{k})\delta(\vec{p}-\vec{k}). \end{aligned} \tag{26}$$

This is the right covariant normalization for relativistic single particle states.

We can make two particle states,

$$|\vec{k},\vec{p}\rangle = a^\dagger(k)a^\dagger(p)|0\rangle. \tag{27}$$

This has

$$P^\mu|k,p\rangle = (k^\mu + p^\mu)|k,p\rangle. \tag{28}$$

so it has the right momentum to be a state of two particles. Note that

$$|k,p\rangle = |p,k\rangle \tag{29}$$

so the particles are bosons. This is built into the theory: *relativistic spin 0 particles are bosons.*

The normalization is

$$\begin{aligned}
\langle p'k'|pk\rangle &= \langle 0|a(p')a(k')a^\dagger(k)a^\dagger(p)|0\rangle \\
&= (2\pi)^3 2\omega(\vec{k})\delta(\vec{k}'-\vec{k})\langle 0|a(p')a^\dagger(p)|0\rangle \\
&\quad +\langle 0|a(p')a^\dagger(k)a(k')a^\dagger(p)|0\rangle \\
&= (2\pi)^3 2\omega(\vec{k})\delta(\vec{k}'-\vec{k})(2\pi)^3 2\omega(\vec{p})\delta(\vec{p}'-\vec{p}) \\
&\quad +(2\pi)^3 2\omega(\vec{k})\delta(\vec{p}'-\vec{k})\langle 0|a(k')a^\dagger(p)|0\rangle \\
&\quad +\langle 0|a^\dagger(k)a(p')a(k')a^\dagger(p)|0\rangle \\
&= (2\pi)^3 2\omega(\vec{k})\delta(\vec{k}'-\vec{k})(2\pi)^3 2\omega(\vec{p})\delta(\vec{p}'-\vec{p}) \\
&\quad +(2\pi)^3 2\omega(\vec{k})\delta(\vec{p}'-\vec{k})(2\pi)^3 2\omega(\vec{p})\delta(\vec{p}-\vec{k}').
\end{aligned} \tag{30}$$

Note carefully that there are two terms here since you can't tell which particle is which.

We can make the same construction for N particle states. We can take linear combinations of N particle states with different momenta. We can take linear combinations of states with different numbers N of particles. The resulting vector space is called Fock space.

Let's now look at the idea of taking linear combinations of 1 particle states with different momenta.The general one particle state is a linear combination

$$|\psi\rangle = (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;\psi(k)|k\rangle. \tag{31}$$

Note that the wave function ψ is

$$\psi(p) = \langle p|\psi\rangle. \tag{32}$$

The normalization integral is

$$\begin{aligned}
\langle \phi|\psi\rangle &= (2\pi)^{-6}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;\int \frac{d\vec{p}}{2\omega(\vec{p})}\;\phi^*(k)\,\psi(p)\langle k|p\rangle \\
&= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;\phi^*(k)\,\psi(k).
\end{aligned} \tag{33}$$

Thus for a normalized state we want

$$(2\pi)^{-3}\int\frac{d\vec{k}}{2\omega(\vec{k})}\ |\psi(k)|^2=1 \tag{34}$$

For an N particle state we would define

$$|\psi\rangle=\frac{1}{N!}\ (2\pi)^{-3N}\left(\prod_i\int\frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)\psi(k_1,\dots k_N)|k_1,\dots k_N\rangle. \tag{35}$$

where ψ is a symmetric function of its momentum arguments. We can work out the normalization integral. There are $N!$ terms, all of them equal. We get

$$1=\langle\psi|\psi\rangle=\frac{1}{N!}\ (2\pi)^{-3N}\left(\prod_i\int\frac{d\vec{k}_i}{2\omega(\vec{k}_i)}\right)|\psi(k_1,\dots k_N)|^2. \tag{36}$$

I have chosen the original normalization so that we have the following interpretation. $|\psi(k_1,\dots k_N)|^2$ is the probability to find one particle with momentum k_1, one with momentum k_2, ..., and one with momentum k_N. (These are probabilities per $d\vec{k}/[(2\pi)^3 2\omega]$). We don't say *which* particle is number 1, which number 2, *etc.*. Thus if we integrate over the whole multiparticle momentum space, we count each distinct configuration of particles $N!$ times. For that reason, the normalization integral has a $1/N!$. These "counting factors" come up frequently in practical problems.

7 Creating discrete states

What if we want to create a state with a normalizable wave function, ψ, as contrasted with a momentum eigenstate. We can use

$$a^\dagger_\psi=(2\pi)^{-3}\int\frac{d\vec{k}}{2\omega(\vec{k})}\ \psi(k)a^\dagger(k), \tag{37}$$

where

$$(2\pi)^{-3}\int\frac{d\vec{k}}{2\omega(\vec{k})}\ |\psi(k)|^2=1. \tag{38}$$

If we apply $a^\dagger_\psi$ to the vacuum, we get the one particle state $|\psi\rangle$ that we want. We have

$$\langle\psi|\psi\rangle=1. \tag{39}$$

For example $|\psi\rangle$ could be the state with one photon in a certain mode of a cavity with walls that reflect photons.

It is of some importance to understand the state with N particles in this same state. This state is

$$|N,\psi\rangle = \frac{1}{\sqrt{N!}}\,(a_\psi^\dagger)^N|0\rangle. \tag{40}$$

Exercise. Prove that this state is normalized to $\langle N,\psi|N,\psi\rangle = 1$.

What if we have an N particle state and we apply $a_\psi^\dagger$ to it. We get

$$\begin{aligned} a_\psi^\dagger|N,\psi\rangle &= \frac{1}{\sqrt{N!}}\,(a_\psi^\dagger)^{N+1}|0\rangle \\ &= \frac{\sqrt{(N+1)}}{\sqrt{(N+1)!}}\,(a_\psi^\dagger)^{N+1}|0\rangle \\ &= \sqrt{(N+1)}\,|N+1,\psi\rangle. \end{aligned} \tag{41}$$

The factor $\sqrt{(N+1)}$ implies that it is easy to make particles if you already have a lot of them. For instance suppose we have an atom with states $|1\rangle$ and $|0\rangle$ and an atom state changing operator A with $A|1\rangle = |0\rangle$, $A|0\rangle = 0$, $A^\dagger|0\rangle = |1\rangle$, and $A^\dagger|1\rangle = 0$. Suppose that the hamiltonian is $H = Aa_\psi^\dagger + A^\dagger a_\psi$. Then

$$|\langle N+1,\psi;0|H|N,\psi;1\rangle|^2 = N+1. \tag{42}$$

The probability that our atom will decay by creating a photon is $N+1$ times bigger if there are already N photons present.

8 Energy of the vacuum

We had

$$H = (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;\omega(\vec{k})\,\tfrac{1}{2}[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})]. \tag{43}$$

Let's rewrite this as

$$\begin{aligned} H &= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;\omega(\vec{k})\,[a^\dagger(\vec{k})a(\vec{k}) + \tfrac{1}{2}(2\pi)^3 2\omega(\vec{k})\delta(\vec{0})]. \\ &= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;\omega(\vec{k})\,a^\dagger(\vec{k})a(\vec{k}) + \tfrac{1}{2}\int d\vec{x}\int \frac{d\vec{k}}{(2\pi)^3}\;\omega(\vec{k}). \end{aligned} \tag{44}$$

Here I have written $(2\pi)^3\delta(\vec{0})$ as an integral over all space of $\exp(i\vec{k}\cdot\vec{x})$ at $\vec{k}=0$. The first term here (sometimes written as $:H:$, the "normal ordered hamiltonian") has the property that $:H:\,|0\rangle = 0$.

The next term is the energy of the vacuum. It is an integral over space of the energy density of the vacuum

$$\mathcal{E}_0 = \tfrac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3}\;\omega(\vec{k}). \tag{45}$$

This is badly divergent. It is the ground state energy for every oscillator that can fit into a little volume of space, and there are oscillators for each wavelength $1/k$. Perhaps in the real universe, this integral is cut off at k of the order of the Planck mass. Then

$$\mathcal{E}_0 \sim M_{\rm Planck}^4. \tag{46}$$

Now some people argue that this doesn't matter. We just subtract $\mathcal{E}_0$ from the hamiltonian. Energy differences detected in experiments will not see any effect. However, $\mathcal{E}_0 = \langle 0|T^{00}|0\rangle$ couples to gravity. If $\mathcal{E}_0$ were this big, the universe would have collapsed in roughly a Planck time and we would not be here. In fact, there is now an approximate measurement of $\mathcal{E}_0$. The result is $(\mathcal{E}_0)^{1/4} \approx 10^{-3}$ eV. Nobody knows where this number comes from.

For the momentum operator, following the same argument, we get

$$\begin{aligned} P^j &= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;k^j\,a^\dagger(\vec{k})a(\vec{k}) + \tfrac{1}{2}\int d\vec{x}\int \frac{d\vec{k}}{(2\pi)^3}\;k^j \\ &= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}\;k^j\,a^\dagger(\vec{k})a(\vec{k}), \end{aligned} \tag{47}$$

where, in the last step, the integral vanishes since it is odd under $\vec{k}\to-\vec{k}$. Thus $P^j|0\rangle = 0$. Thus the problem is with the energy density of the vacuum, but not with the momentum density.