Quantum Electrodynamics[1]

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1 The action

We begin with an argument that quantum electrodynamics is a natural extension of the theory of a free Dirac field, with action

$$S[\bar\psi,\psi] = \int d^4x\ \bar\psi(x)\{i\partial\!\!\!/ - m\}\psi(x). \tag{1}$$

Notice that this action is invariant under the transformation

$$\begin{aligned}\psi(x) &\to e^{-iQe\alpha}\psi(x)\\ \bar\psi(x) &\to \bar\psi(x)e^{iQe\alpha}\end{aligned} \tag{2}$$

Here Qe is a constant that tells us how much to rotate the field ψ under the rotation specified by the parameter α. We sill later identify $e = +|e|$ with the proton electric charge. Then Qe will be the charge of the particle annihilated by the field ψ. That is, $Q = -1$ for an electron. This notation allows us to describe several fields, ψ_J, each of which transforms as above with charge $Q_J e$. Note that Peskin and Schroeder use the symbol e to mean $-|e|$. I think that's confusing.

The action is *not* invariant under the transformation

$$\begin{aligned}\psi(x) &\to e^{-iQe\alpha(x)}\psi(x)\\ \bar\psi(x) &\to \bar\psi(x)e^{iQe\alpha(x)}\end{aligned} \tag{3}$$

in which α depends on the space-time point x. This is called a gauge thansformation. In fact, we have

$$S[\bar\psi,\psi] \to \int d^4x\ \bar\psi(x)\{i\partial\!\!\!/ + Qe\frac{\partial\alpha(x)}{\partial x^\mu}\,\gamma^\mu - m\}\psi(x). \tag{4}$$

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Let us try to extend tha theory so that it is invariant under gauge transformations. We add a field $A_\mu(x)$ with a transformation law

$$A_\mu(x) \to A_\mu(x) + \frac{\partial\alpha(x)}{\partial x^\mu}. \tag{5}$$

We take

$$S[\bar\psi, \psi, A] = \int d^4x\ \bar\psi(x)\{i\partial\!\!\!/ - Qe A\!\!\!/(x) - m\}\psi(x) + \cdots . \tag{6}$$

The $+\cdots$ indicates that we are going to have to add something else. But we can note immediately that the terms we have so far are invariant under gauge thransformations.

We need to add a "kinetic energy" term for A_μ, something analogous to the integral of $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ for a scalar field. Whatever we add should be gauge invariant by itself. We take

$$S[\bar\psi, \psi, A] = \int d^4x \left[\bar\psi(x)\{i\partial\!\!\!/ - Qe A\!\!\!/(x) - m\}\psi(x) - \tfrac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x)\right], \tag{7}$$

where $F_{\mu\nu}(x)$ is a convenient shorthand for

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{8}$$

We see that $F_{\mu\nu}(x)$ is gauge invariant, so the added term is gauge invariant also.

From this action, we get the equation of motion for ψ:

$$\{i\partial\!\!\!/ - Qe A\!\!\!/(x) - m\}\psi(x) = 0, \tag{9}$$

which is the Dirac equation for an electron in a potential $A_\mu(x)$. The equation of motion of A_μ is

$$\partial_\mu F^{\mu\nu}(x) = J^\nu(x), \tag{10}$$

where

$$J^\nu(x) = Qe\bar\psi(x)\gamma^\nu\psi(x). \tag{11}$$

This is the inhomogenious parts of the Maxwell equations for the electromagnetic fields produced by a current $J^\nu(x)$. The homogenious part of the Maxwell equations follows automatically because $F_{\mu\nu}(x)$ is expressed in terms of the potential $A_\mu(x)$.

Exercise: Derive the equation of motion of A_μ.

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2 Coulomb gauge

We want to use $A^\mu(x)$ as the canonical coordinates for the electromagnetic field. However, two of the four degrees of freedom per space point are illusory because of gauge invariance. For instance, if you had one solution of the eequations of motion based on initial conditions at $t = 0$, you could always change if for $t > t_1 > 0$ by making a gauge thansformation. For this reason, we will choose a gauge for the quantum theory, namely Coulomb gauge. Note that Peskin and Schroeder don't cover Coulomb gauge. However, using the Coulomb gauge is the best way to do the quantization, at least if you don't want to use the path integral formulation of the quantum field theory. Also, Coulomb gauge is important for understanding nonrelativistic quantum mechanics with photons.

The Maxwell equations are

$$\partial_\mu \partial^\mu A^\nu(x) - \partial^\nu \partial_\mu A^\mu(x) = J^\nu(x). \tag{12}$$

We choose Coulomb gauge,

$$\vec\nabla \cdot \vec A(x) = 0. \tag{13}$$

Then the Maxwell equation for A^0 is

$$-\vec\nabla^2 A^0(x) = J^0(x). \tag{14}$$

The solution of this is

$$A^0(\vec x, t) = \int d\vec y\, \frac{1}{4\pi}\, \frac{1}{|\vec x - \vec y|}\; J^0(\vec y, t). \tag{15}$$

We can thus view A^0 not as an independent dynamical variable, but as a constrained variable that simply stands as an abbreviation for the right hand side of Eq. (15).

The equations of motion for the space-components of A^μ are

$$\partial_\mu \partial^\mu A^j(x) = J^j_T(x), \tag{16}$$

where

$$J^j_T(x) = J^j(x) - \partial_j \partial_0 A^0(x). \tag{17}$$

That is

$$J^j_T(\vec x, t) = J^j(\vec x, t) - \partial_j \int d\vec y\, \frac{1}{4\pi}\, \frac{1}{|\vec x - \vec y|}\; \partial_0 J^0(\vec y, t). \tag{18}$$

The subscript T here stands for "transverse" and refers to the fact that $\vec{\nabla}\cdot\vec{J}_T = 0$, so that in momentum space, $\vec{J}_T = 0$ is transverse to the momentum vector $\vec{k}$. This follows if we use the equation of motion for the Dirac field, which implies $\partial_0 J^0(x) = -\partial_i J^i(x)$. Then

$$J_T^j(\vec{x},t) = J^j(\vec{x},t) + \partial_j \int d\vec{y}\ \frac{1}{4\pi}\ \frac{1}{|\vec{x}-\vec{y}|}\ \partial_i J^i(\vec{y},t) \tag{19}$$

A better way to write J_T is

$$J_T^i(\vec{x},t) = \int d\vec{y}\ \delta_T^{ij}(\vec{x}-\vec{y})\ J^j(\vec{y},t), \tag{20}$$

where

$$\delta_T^{ij}(\vec{x}-\vec{y}) = \int \frac{d\vec{k}}{(2\pi)^3}\ e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right). \tag{21}$$

In this form, it is evident that $\partial_i J_T^i(\vec{x},t) = 0$. Eq. (20) follows from Eq. (19) by using

$$\int \frac{d\vec{k}}{(2\pi)^3}\ e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\ \frac{1}{\vec{k}^2} = \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}. \tag{22}$$

We need the lagrangian. Using the Coulomb gauge condition and integrating by parts, we get

$$\begin{aligned} L \ =\ \int d\vec{x}\Big\{ & \tfrac{1}{2}(\partial_0 A^j)(\partial_0 A^j) - \tfrac{1}{2}(\partial_i A^j)(\partial_i A^j) \\ & + \tfrac{i}{2}\bar{\psi}\gamma^0(\partial_0\psi) - \tfrac{i}{2}(\partial_0\bar{\psi})\gamma^0\psi + \bar{\psi}\{i\partial_j\gamma^j - m\}\psi \\ & - A^0 J^0 + \vec{A}\cdot\vec{J} - \tfrac{1}{2}A^0\vec{\nabla}^2 A^0 \Big\}. \end{aligned} \tag{23}$$

Since $-\vec{\nabla}^2 A^0 = J^0$ this is

$$\begin{aligned} L \ =\ \int d\vec{x}\Big\{ & \tfrac{1}{2}(\partial_0 A^j)(\partial_0 A^j) - \tfrac{1}{2}(\partial_i A^j)(\partial_i A^j) \\ & + \tfrac{i}{2}\bar{\psi}\gamma^0(\partial_0\psi) - \tfrac{i}{2}(\partial_0\bar{\psi})\gamma^0\psi + \bar{\psi}\{i\partial_j\gamma^j - m\}\psi \\ & - \tfrac{1}{2}A^0 J^0 + \vec{A}\cdot\vec{J} \Big\}. \end{aligned} \tag{24}$$

From this we compute the hamiltonian,

$$\begin{aligned} H \ =\ \int d\vec{x}\Big\{ & \tfrac{1}{2}\pi^j\pi^j + \tfrac{1}{2}(\partial_i A^j)(\partial_i A^j) \\ & - \bar{\psi}\{i\partial_j\gamma^j - m\}\psi \\ & + \tfrac{1}{2}A^0 J^0 - \vec{A}\cdot\vec{J} \Big\}. \end{aligned} \tag{25}$$

where $\pi^j(x)$ is the canonical momentum conjugate to the field $A^j(x)$,

$$\pi^j(x) = \frac{\partial}{\partial t} A^j(x). \tag{26}$$

Since $\partial_j A^j = 0$, we have to impose $\partial_j \pi^j = 0$. Also, remember that A^0 here is just an abbreviation for the potential produced by a charge density J^0. Thus $\int d\vec{x}\, \frac{1}{2} A^0 J^0$ is the Coulomb energy of the charge distribution J^0. The $\vec{A} \cdot \vec{J}$ term is the interaction by which moving charges make photons.

We will need commutation/anticommutation relations. For the Dirac field, we know what to do:

$$\begin{aligned} \{\psi(\vec{x},t), \psi(\vec{y},t)\} &= 0 \\ \{\psi(\vec{x},t), \bar{\psi}(\vec{y},t)\} &= \gamma^0\, \delta(\vec{x}-\vec{y}). \end{aligned} \tag{27}$$

Exercise: Show that with these anticommutation relations, commuting the Hamiltonian with ψ gives the Dirac equation for ψ.

For the vector potential, we would be tempted to let $[A^i(\vec{x},t), \pi^j(\vec{y},t)]$ be $i\delta_{ij}$ times a delta function. However, that won't work because it is inconsistent with the Coulomb gauge condition. So try

$$\begin{aligned} [A^i(\vec{x},t), A^j(\vec{y},t)] &= 0 \\ [\pi^i(\vec{x},t), \pi^j(\vec{y},t)] &= 0 \\ [A^i(\vec{x},t), \pi^j(\vec{y},t)] &= i\, \delta_T^{ij}(\vec{x}-\vec{y}). \end{aligned} \tag{28}$$

This is essentially a delta function for the transverse degrees of freedom.

Exercise: Show that with these commutation relations, commuting the Hamiltonian with A^j and π^j gives the Maxwell equations for A^j.

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3 Free photons

If we remove the interactions with electrons, the equation of motion for the vector potential in Coulomb gauge

$$\partial_j A^j(x) = 0, \tag{29}$$

is

$$\partial_\mu \partial^\mu A^j(x) = 0. \tag{30}$$

With no charge around, $A^0(x) = 0$. We can easily solve this:

$$A^\mu(x) = (2\pi)^{-3} \int \frac{d\vec{k}}{2\omega(\vec{k})} \sum_{s=-1,1} \left\{ e^{-ik\cdot x} \epsilon^\mu(k,s) a(k,s) + e^{ik\cdot x} \epsilon^\mu(k,s)^* a^\dagger(k,s) \right\}. \tag{31}$$

Here $k^0 = \omega(\vec{k}) \equiv |\vec{k}|$. That gives $\partial_\mu \partial^\mu A^j(x) = 0$. To get $\partial_j A^j(x) = 0$ we need

$$k^j \epsilon^j(k,s) = 0. \tag{32}$$

There are two solutions of this, which we can take to be the solutions with helicity $+1$ and -1. For $\vec{k}$ equal to a reference momentum $\vec{k}_0$ along the z axis, these are, written as four vectors $(\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3)$,

$$\begin{aligned} \epsilon(\vec{k}_0, +1) &= \frac{1}{\sqrt{2}}(0,1,i,0) \\ \epsilon(\vec{k}_0, -1) &= \frac{1}{\sqrt{2}}(0,-1,i,0). \end{aligned} \tag{33}$$

These describe left circularly polarized and right circularly polarized photons, respectively. For any other momentum, we can boost along the z axis enough to chagne $|\vec{k}_0|$ to $|\vec{k}|$. This leaves ϵ^μ unchanged. Then we can rotate around the $\vec{k}_0 \times \vec{k}$ axis by the angle between $\vec{k}_0$ and $\vec{k}$. That is, we apply the Wigner construction that we learned about last fall. This gives us polarization vectors ϵ with normalization

$$\epsilon(\vec{k}, s)^\mu \, \epsilon(\vec{k}, s')^*_\mu = -\delta_{ss'} \tag{34}$$

They also obey the spin sum relation

$$\sum_s \epsilon(\vec{k}, s)^i \, (\epsilon(\vec{k}, s)^j)^* = \delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \equiv P^{ij}(\vec{k}). \tag{35}$$

Proof: the vectors $\epsilon(k,s)$ are a orthogonal normalized basis for the space of vectors $\vec{\epsilon}$ with $\vec{k} \cdot \vec{\epsilon} = 0$, so the matrix on the left constructed from the ϵ vectors is the projection onto this space. But that's what the matrix on the right is.

The coefficients are the photon annihilation operators $a(k,s)$ and creaton operators $a^\dagger(k,s)$. They have the commutation relations

$$\begin{aligned}
[a(k,s),a(k',s')] &= 0 \\
[a^\dagger(k,s),a^\dagger(k',s')] &= 0 \\
[a(k,s),a^\dagger(k',s')] &= \delta_{ss'}(2\pi)^3 2\omega(\vec{k})\delta(\vec{k}-\vec{k}'). \qquad (36)
\end{aligned}$$

Exercise: Prove that these commutation relations in momentum space lead to the desired equal time commutation relations for the fields $\vec{A}(x)$ and $\vec{\pi}(x)$.

Now let's work out the vacuum expectation value of the time ordered product of two $\vec{A}$ fields, which we know will be the photon propagator in the interacting theory. We seek

$$\langle 0|TA^i(x)A^j(0)|0\rangle = D_F^{ij}(x) \qquad (37)$$

For $x^0 > 0$ we have

$$\begin{aligned}
&D_F^{ij}(x) \\
&= \langle 0|A^i(x)A^j(0)|0\rangle \\
&= (2\pi)^{-6}\int \frac{d\vec{k}}{2\omega(\vec{k})}e^{-ik\cdot x}\int \frac{d\vec{p}}{2\omega(\vec{k})}\sum_{ss'}\epsilon^i(k,s)\epsilon^j(p,s')^*\langle 0|a(k,s)\,a^\dagger(p,s')|0\rangle \\
&= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}e^{-ik\cdot x}\sum_s \epsilon^i(k,s)\epsilon^j(k,s)^* \\
&= (2\pi)^{-3}\int \frac{d\vec{k}}{2\omega(\vec{k})}e^{-ik\cdot x}P^{ij}(\vec{k}) \\
&= (2\pi)^{-4}\int d^4k\ e^{-ik\cdot x}\ \frac{2\pi}{2\omega(\vec{k})}\delta(k^0-\omega(\vec{k}))P^{ij}(\vec{k}) \\
&= (2\pi)^{-4}\int d^4k\ e^{-ik\cdot x}\ \frac{P^{ij}(\vec{k})}{2\omega(\vec{k})}\left\{\frac{i}{k^0-\omega(\vec{k})+i\epsilon}-\frac{i}{k^0-\omega(\vec{k})-i\epsilon}\right\} \\
&= (2\pi)^{-4}\int d^4k\ e^{-ik\cdot x}\ \frac{P^{ij}(\vec{k})}{2\omega(\vec{k})}\left\{\frac{i}{k^0-\omega(\vec{k})+i\epsilon}-\frac{i}{k^0+\omega(\vec{k})-i\epsilon}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-4} \int d^4k \; e^{-ik\cdot x} \; \frac{P^{ij}(\vec{k})}{2\omega(\vec{k})} \left\{ \frac{2i\omega(\vec{k})}{(k^0)^2 - \omega(\vec{k})^2 + i\epsilon} \right\} \\
&= (2\pi)^{-4} \int d^4k \; e^{-ik\cdot x} \; \frac{iP^{ij}(\vec{k})}{k^2 + i\epsilon}
\end{aligned} \tag{38}
$$

In the third line from the last, we changed the sign in front of ω in the denominator. This did not affect the result because, given the sign of the $i\epsilon$ in this term, the term does not contribute for $x^0 > 0$. Now we do a similar calculation for $x^0 < 0$, with the same result. This gives us the rule for a photon propagator in momentum space:

$$
\frac{iP^{ij}(\vec{k})}{k^2 + i\epsilon}. \tag{39}
$$

4 Feynman rules in Coulomb gauge

We thus get the Feynman rules in Coulomb gauge. First, to calculate an S-matrix element $i\mathcal{M}$ we need an amputated Green function times

- $\sqrt{Z}_\psi \mathcal{U}(k, x)$ for an incoming electron.
- $\sqrt{Z}_\psi \bar{\mathcal{U}}(k, x)$ for an outgoing electron.
- $\sqrt{Z}_\psi \mathcal{V}(k, x)$ for an outgoing positron.
- $\sqrt{Z}_\psi \bar{\mathcal{V}}(k, x)$ for an incoming positron.
- $\sqrt{Z}_A \epsilon^\mu(k, s)$ for an incoming photon.
- $\sqrt{Z}_A \epsilon^\mu(k, s)^*$ for an outgoing photon.

The factors $\sqrt{Z}$ are to be calulated perturbatively. The electron propagator as one approaches the single particle pole is

$$
S_F \sim \frac{iZ_\psi [\not{k} + M]}{k^2 - M^2 + i\epsilon} \tag{40}
$$

where M is the physical electron mass. This defines Z_ψ. The photon propagator as one approaches the single photon pole is

$$
D_F \sim \frac{iZ_A P^{ij}(k)}{k^2 + i\epsilon}. \tag{41}
$$

(The physical photon has mass zero.) This defines Z_A.

Now to calculate a perturbative contributi0n to the Green function, we use the rules

- Label the lines by their momenta and energy, using momentum and energy conservation at each vertex.
- For each loop, there will be one energy and momentum that is not constrained by energy and momentum conservation. Supply an integration

$$(2\pi)^{-4}\int d^4p. \tag{42}$$

- To each electron line associate a propagator

$$\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} \tag{43}$$

where k^μ is the momentum in the direction of the electron-number arrow.

- To each transversely polarized photon line associate a propagator

$$\frac{iP^{\mu\nu}(k)}{k^2+i\epsilon} \tag{44}$$

where $P^{0\mu}=P^{\mu 0}=0$ and

$$P^{ij}(k)=\delta^{ij}-\frac{k^ik^j}{\vec{k}^2} \tag{45}$$

- To each coulomb photon line associate a "propagator"

$$\frac{in^\mu n^\nu}{\vec{k}^2} \tag{46}$$

where $n=(1,0,0,0)$ is a unit vector in the time direction.

- For each vertex representing an electron-photon interaction associate a factor

$$-iQg\gamma_\mu \tag{47}$$

where Q is the charge of the electron in units of e (ie $Q=-1$).

- There is a minus sign between graphs that are identical except for exchange of two fermion lines.

- There is a minus sign for each fermion loop.

Notice that in each place where we can exchange a transverse photon, we can also exchange a Coulomb photon. The combined photon propagator is

$$D_F^{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon} \left\{ P^{\mu\nu}(k) + \frac{k^2}{\vec{k}^2} \, n^\mu n^\mu \right\}. \tag{48}$$

The k^2 in the second term cancels the $1/k^2$ and tells us that the Coulomb photon doesh't really propagate. We can rewrite the photon propagator in a useful way as follows, letting $\tilde{k}^\mu$ be a vector consisting of only the space-parts of k^μ, that is $\tilde{k} = (0, k^1, k^2, k^3)$

$$\begin{aligned} D_F^{\mu\nu}(k) &= \frac{i}{k^2 + i\epsilon} \left\{ -g^{\mu\nu} + n^\mu n^\nu - \frac{\tilde{k}^\mu \tilde{k}^\nu}{\vec{k}^2} + \frac{(k^0)^2 - \vec{k}^2}{\vec{k}^2} \, n^\mu n^\mu \right\} \\ &= \frac{i}{k^2 + i\epsilon} \left\{ -g^{\mu\nu} - \frac{\tilde{k}^\mu \tilde{k}^\nu}{\vec{k}^2} + \frac{(k^0 n^\mu)(k^0 n^\nu)}{\vec{k}^2} \right\} \\ &= \frac{i}{k^2 + i\epsilon} \left\{ -g^{\mu\nu} - \frac{\tilde{k}^\mu \tilde{k}^\nu}{\vec{k}^2} + \frac{(k^\mu - \tilde{k}^\mu)(k^\nu - \tilde{k}^\nu)}{\vec{k}^2} \right\} \\ &= \frac{i}{k^2 + i\epsilon} \left\{ -g^{\mu\nu} + \frac{k^\mu k^\nu - k^\mu \tilde{k}^\nu - \tilde{k}^\mu k^\nu}{\vec{k}^2} \right\} \end{aligned} \tag{49}$$

5 From Coulomb gauge to Feynman gauge

We see that there are terms in the photon propagator in Coulomb gauge that have either a k^μ or a k^ν. What happens when a term with a k^μ connects to a corresponding interaction J_μ? We get a structure of the form

$$\frac{i[\not{k}_1 + m]}{k_1^2 - m^2 + i\epsilon} \, (-ieQ)(\not{k}_1 - \not{k}_2) \, \frac{i[\not{k}_2 + m]}{k_2^2 - m^2 + i\epsilon} \tag{50}$$

or

$$\frac{i[\not{k}_1 + m]}{k_1^2 - m^2 + i\epsilon} \, (-ieQ)([\not{k}_1 - m] - [\not{k}_2 - m]) \, \frac{i[\not{k}_2 + m]}{k_2^2 - m^2 + i\epsilon} \tag{51}$$

Using $\not k \not k = k^2$, this is

$$(-ieQ)\frac{i[k_1^2 - m^2]}{k_1^2 - m^2 + i\epsilon}\,\frac{i[\not k_2 + m]}{k_2^2 - m^2 + i\epsilon} \;-\; (-ieQ)\frac{i[\not k_1 + m]}{k_1^2 - m^2 + i\epsilon}\,\frac{i[k_2^2 - m^2]}{k_2^2 - m^2 + i\epsilon} \tag{52}$$

That is

$$eQ\frac{i[\not k_2 + m]}{k_2^2 - m^2 + i\epsilon} \;-\; eQ\frac{i[\not k_1 + m]}{k_1^2 - m^2 + i\epsilon}. \tag{53}$$

That is, there are two terms, with opposite signs. In each term, a propagator is eliminated and the photon momentum is delivered at the next vertex in the larger graph – that vertex that was next to the propagator that got eliminated.

Let's apply this to graphs for a full (i.e. not amputated) Green function. Pick one of the exchanged photons and consider one of the k^μ terms in its propagator. Sum over all the graphs in which this photon attaches to various places on the electron line. If the electron line was a loop, all of the contributions cancel. If it led from one external point to another, all of the terms cancel except for the two in which the new special interaction is at one or the other of the electron line. Now for the scattering matrix by taking the residue at the poles representing propagation of electrons into the final state or from the initial state. Except for the case of self-energy graphs on the external lines, the extra terms do not have these poles. That is, they are present in Green functions, but not in the scattering matrix. The effect of self-energy graphs takes a little analysis to work out: we have to analyze the $\sqrt{Z}$ factors. We will leave this for later. The net effect is that the k^μ terms all cancel in a calculation of the S-matrix.

Thus if we are concerned with the scattering matrix, we can take the photon propagator to be

$$\frac{-ig^{\mu\nu}}{k^2 + i\epsilon} \tag{54}$$

If we do this, we say that we are calculating in Feynman gauge. It is a lot simpler for practical calculations than Coulomb gauge.

6 A cross section

Let's see how to calculate a cross section. A good example is $e^+ + e^- \to \mu^+ + \mu^-$. Recall that the relation between cross sections and the s-matrix is as follows. Using a covariant normalization of states, we define the "invariant amplitude " $\mathcal{M}$ by

$$
\begin{aligned}
{}_{\rm out}\langle p_1, p_2, \ldots | k_A, k_B\rangle_{\rm in} &= \langle p_1, p_2, \ldots | k_A, k_B\rangle \\
&\quad +(2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f)\; i\mathcal{M}.
\end{aligned}
\tag{55}
$$

Then,

$$
d\sigma = \left(\prod_f \frac{d\vec{p}_f}{(2\pi)^3\, 2p_f^0}\right) \frac{1}{2p_A^0 2p_B^0 |v_A - v_B|}\,(2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f)\; |\mathcal{M}|^2.
\tag{56}
$$

This relation is nicely proved in Peskin and Scrhoeder Section 4.5. You should study this proof carefully.

This is for some fixed values of the spins. If the initial state particles are unpolarized (*i.e.* an average over the possible spin states) then

- Average over the initial state spins. That is, for either an initial state electron or positron or an initial state photon, we insert

$$
\frac{1}{2}\sum_s
\tag{57}
$$

If the spins of the final state particles are not observed then

- Sum over the initial state spins. That is, for either an final state electron or positron or a final state photon, we insert

$$
\sum_s
\tag{58}
$$

There are some nice simplifications available:

- For initial state or final state electrons, we can use

$$
\sum_s \mathcal{U}(p,s)\bar{\mathcal{U}}(p,s) = \not{p} + m.
\tag{59}
$$

- For initial state or final state positrons, we can use

$$
\sum_s \mathcal{V}(p,s)\bar{\mathcal{V}}(p,s) = \not{p} - m.
\tag{60}
$$

- For initial state or final state photons, we can use

$$\sum_s \epsilon^\mu(p,s)\epsilon^\nu(p,s)^* = P_T^{\mu\nu}(p). \tag{61}$$

Furthermore, we can write this as (for $p^2 = 0$)

$$-g^{\mu\nu} + \frac{p^\mu p^\nu - p^\mu \tilde{p}^\nu - \tilde{p}^\mu p^\nu}{\vec{p}^2} \tag{62}$$

The p^μ and p^ν terms will not contribute because of the gauge invariance identities. Thus we can replace

$$\sum_s \epsilon^\mu(p,s)\epsilon^\nu(p,s)^* \to -g^{\mu\nu}. \tag{63}$$

Exercise: Use these rules and the gamma matrix identities from the next section to calculate the cross section $d\sigma/d\cos\theta$ for the process $e^+ + e^- \to \mu^+ + \mu^-$ in the c.m. frame. To make this as simple as possible, you may take the masses of both the electron and the muon to be zero.

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7 Some identies

If we are summing over spins of electrons or other spin 1/2 particles, then our expression for a cross section will involve traces of Dirac matrices. These can be calculated by using the rules that follow. We write the rules for d dimensions of space-time because when we deal with divergent graphs we will use a trick that involves doing the algebra in d dimensions with $d \neq 4$. We will always take

$$Tr[1] = 4 \tag{64}$$

for the trace of the unit matrix in Dirac spinor space. Also, we will take

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{65}$$

However, in d dimensions,

$$g^\mu_\mu = \sum_{\mu=1}^{d} 1 = d. \tag{66}$$

Here are the rules for the trace of 0, 2, or 4 gamma matrices:

$$\begin{aligned} \mathrm{Tr}[1] &= 4 \\ \mathrm{Tr}[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\ \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4\,[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}] \end{aligned} \tag{67}$$

Exercise: The first of these rules just gives the dimensionality of the Dirac matrices. Use the anticommutation relations of Dirac matrices to establish the other two rules.

The trace of an odd number of gamma matrices vanishes for $d = 4$. For $d \neq 4$ we can take it as a definition that the trace of an odd number of gamma matrices vanishes.

Exercise: Prove this for $d = 4$. Use the fact that there is a matrix, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, that anticommutes with the γ^μ.

You will often meet traces containing structures like $\gamma^\mu\gamma^\alpha\gamma_\mu$. These can be simplified using

$$\begin{aligned} \gamma^\mu\gamma_\mu &= d \\ \gamma^\mu\gamma^\alpha\gamma_\mu &= -(d-2)\gamma^\alpha \\ \gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu &= 4g^{\alpha\beta} - (4-d)\gamma^\alpha\gamma^\beta \\ \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\mu &= -2\gamma^\gamma\gamma^\beta\gamma^\alpha + (4-d)\gamma^\alpha\gamma^\beta\gamma^\gamma \end{aligned} \tag{68}$$

Exercise: Prove these.

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8 QED for nonrelativistic particles

In this section we examine quantum electrodynamics for nonrelativistic. This is interesting in its own right because it relates to atomic and condensed matter physics. We will also pay special attention to the interaction of a nonrelativistic particle with a magnetic field. This will prepare us for a calculation of the magnetic moment of the electron in relativistic quantum electrodynamics.

We begin with a free nonrelativistic particle with action

$$S = \int d^4x\ \psi^\dagger_\alpha(x) \left[i\frac{\partial}{\partial t} + \frac{1}{2m}\vec{\nabla}^2 \right] \psi_\alpha(x). \tag{69}$$

Here α is the spin index for the particle field. For a spin 1/2 particle, it takes the values 1/2 and $-1/2$. We can make this into a gauge invariant theory with an electromagnetic field by changing it to

$$\begin{aligned} S &= \int d^4x \Big\{ \psi^\dagger_\alpha(x) \left[i\frac{\partial}{\partial t} - QeA^0 - \frac{1}{2m}(-i\vec{\nabla} - Qe\vec{A})^2 \right] \psi_\alpha(x). \\ &\quad -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \Big\}. \end{aligned} \tag{70}$$

This corresponds to interactions

- Particle interaction with a Coulomb photon:

$$-ieQ\delta_{\alpha\beta} \tag{71}$$

  This includes a unit matrix $\delta_{\alpha\beta}$ acting on the particle spin space.

- Particle interaction with a transverse photon

$$i\,\frac{Qe}{2m}(p_1^j + p_2^j)\,\delta_{\alpha\beta} \tag{72}$$

  where j is the vector index for the photon and $\vec{p}_1$ and $\vec{p}_2$ are the particle momenta before and after the interaction.

- Particle interaction with two transverse photons

$$-i\,\frac{Q^2e^2}{2m}\delta^{ij}\,\delta_{\alpha\beta} \tag{73}$$

Note that there is no interaction with the particle spin.

But there could be an interaction with the particle spin. We are familiar in classical electrodynamics with having a term in the hamiltonian $-\vec{\mu}\cdot\vec{B}$, where $\vec{\mu}$ is the particle's magnetic moment. For a quantum particle, the magnetic moment can be a constant times the particle spin $\vec{S}$:

$$\vec{\mu} = g\,\frac{eQ}{2m}\ \vec{S} \tag{74}$$

The form of the proportionality constant here is motivated considering a particle made up of some classical stuff that has a density $\rho_m(x)$, a velocity $\vec{v}(x)$, and a charge density $(Qe/m)\rho_m(x)$

$$\begin{aligned}\vec{\mu} &= \int d\vec{x}\; \tfrac{1}{2}\,\vec{x} \times (Qe/m)\rho_m(x)\vec{v}(x) \\ &= \frac{Qe}{2m}\,\vec{S}.\end{aligned} \tag{75}$$

where $\vec{S}$ is the angular momentum of the system. This has the form of Eq. (74) with $g = 1$. The "g factor" then describes how the real particle differs from the classical expectation.

We can introduce the magnetic interaction by adding a term to the action of the form

$$\Delta S = \int d^4x\; g\,\frac{Qe}{2m}\; \psi^\dagger_\alpha(x) S^i_{\alpha\beta}\psi_\beta(x)\, B^i(x). \tag{76}$$

Here

$$B^i = \epsilon^{ijk}\partial_j A^k \tag{77}$$

is the magnetic field and $S^j_{\alpha\beta}$ is the matrix representing the particle spin, which for spin 1/2 is

$$S^j_{\alpha\beta} = \tfrac{1}{2}\sigma^j_{\alpha\beta}. \tag{78}$$

This gives an additional interaction vertex in the Feynman rules,

- Magnetic moment interaction with a transverse photon

$$-g\,\frac{Qe}{2m}\, S^i_{\alpha\beta}\epsilon^{ijk}q^j \tag{79}$$

  where k is the vector index for the photon and $\vec{q}$ is the momentum carried into the vertex by the photon.

9 Electric and magnetic form factors

In this section, we find out how to define the electric and magnetic form factors of the electron. This tells us how to define the electron's magnetic moment. Then we will be in a position to calculate the magnetic moment at one loop order.

The electromagnetic current operator is

$$J^\mu(x) = Qe\,\bar{\psi}(x)\gamma^\mu\psi(x). \tag{80}$$

The amplitude for an electron to scatter from an electromagnetic field can be written as

$$\langle k', s'|J^\mu(0)|k, s\rangle = Qe\,\bar{\mathcal{U}}(k's')\Gamma^\mu(k',k)\mathcal{U}(k,s) \tag{81}$$

Here $\Gamma^\mu(k',k)$ is the sum of all graphs for electron scattering by absorbing a single photon. There are some subtleties with this that have to do with renormalization and with infrared divergences, but we will ignor the subleties for now.

We write this in the form

$$\begin{aligned} &Qe\bar{\mathcal{U}}(k's')\Gamma^\mu(k',k)\mathcal{U}(k,s) \\ &\quad = \bar{\mathcal{U}}(k's')\left\{\gamma^\mu QeF_1(q^2) + i\frac{Qe}{2m}\,\sigma^{\mu\nu}q_\nu\,\kappa\,F_2(q^2)\right\}\mathcal{U}(k,s) \end{aligned} \tag{82}$$

Here

$$\sigma^{\mu\nu} = \tfrac{i}{2}\,[\gamma^\mu,\gamma^\nu]. \tag{83}$$

The functions $F_1(q^2)$ and $F_2(q^2)$ are Lorentz scalar functions of q^2. They are called "form factors." They obey

$$F_1(0) = F_2(0) = 1 \tag{84}$$

The condidition $F_1(0)$ defines Qe. The condition $F_2(0)$ defines κ, which is called the anomalous magnetic moment of the electron.

The two terms displayed are evidently Lorentz invariant and parity invarient. They also obey current conservation (or gauge invariance) in the sense that

$$q_\mu\langle k', s'|J^\mu(0)|k, s\rangle = 0. \tag{85}$$

These two are the only possibilities. First, the possible gamma matrix structures are

$$1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu} \tag{86}$$

where $\gamma_5 = (-i/4!)\epsilon^{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta$. (There are sixteen of these and the space of gamma matrices is sixteen dimensional.) Then for Γ^μ

- $1 \times (k^\mu + k'^\mu)$ is OK but is equivalent to the terms we have, as we will see.

- q^μ violates current conservation.

- $\gamma_5(k^\mu + k'^\mu)$ violates parity.
- $\gamma_5 q^\mu$ violates current parity.
- γ^μ is OK and we have it.
- $\gamma^\mu\gamma_5$ violates parity.
- $\sigma^{\mu\nu}q_\nu$ is OK and we have it.
- $\sigma^{\mu\nu}(k_\nu + k'_\nu)$ violates current conservation.

To show that a term $1 \times (k^\mu + k'^\mu)$ is equivalent to the terms we have already, we use what is called the Gordon identity:

$$\bar{\mathcal{U}}(k's')\gamma^\mu\mathcal{U}(k,s) = \frac{1}{2m}\bar{\mathcal{U}}(k's')\left\{(k^\mu + k'^\mu) + i\sigma^{\mu\nu}q_\nu\right\}\mathcal{U}(k,s) \qquad (87)$$

This is easily proved starting by defining $p^\mu = (k'^\mu + k^\mu)$ and writing

$$\begin{aligned} \bar{\mathcal{U}}(k's')2m\gamma^\mu\mathcal{U}(k,s) &= \bar{\mathcal{U}}(k's')\{\not{k}'\gamma^\mu + \gamma^\mu\not{k}\}\mathcal{U}(k,s) \\ &= \tfrac{1}{2}\bar{\mathcal{U}}(k's')\{(\not{p} + \not{q})\gamma^\mu + \gamma^\mu(\not{p} - \not{q})\}\mathcal{U}(k,s) \\ &= \tfrac{1}{2}\bar{\mathcal{U}}(k's')\{p_\nu\{\gamma^\mu, \gamma^\nu\} - q_\nu[\gamma^\mu, \gamma^\nu]\}\mathcal{U}(k,s) \\ &= \tfrac{1}{2}\bar{\mathcal{U}}(k's')\{2p_\mu + 2iq_\nu\sigma^{\mu\nu}\}\mathcal{U}(k,s). \end{aligned} \qquad (88)$$

Now let's see how this general form works in the non-relativistic limit. First, we can use the Gordon identity to rewrite it as

$$\begin{aligned} &Qe\bar{\mathcal{U}}(k's')\Gamma^\mu(k',k)\mathcal{U}(k,s) \qquad (89) \\ &= \bar{\mathcal{U}}(k's')\left\{\frac{(k^\mu + k'^\mu)}{2m}QeF_1(q^2) + i\frac{Qe}{2m}\,\sigma^{\mu\nu}q_\nu\left[F_1(q^2) + \kappa\, F_2(q^2)\right]\right\}\mathcal{U}(k,s). \end{aligned}$$

Now we need the Dirac spinors in the nonrelativistic limit. We use gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad (90)$$

Then the free Dirac equation for

$$\mathcal{U} = \sqrt{E+m}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad (91)$$

is

$$\begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \tag{92}$$

This gives

$$\eta = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\xi \tag{93}$$

Thus for $p \ll m$, η is small compared to ξ. (With the definitions given above, our standard normalization $\bar{\mathcal{U}}\mathcal{U} = 2m$ corresponds to $\xi^\dagger\xi = 1$.)

The matrices $\sigma^{\mu\nu}$ are

$$\sigma^{0j} = \begin{pmatrix} 0 & i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} \qquad \sigma^{ij} = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{94}$$

Note that σ^{0j} gives $\xi \times \eta$ terms, which are small compared to $\xi \times \xi$ terms in the nonrelativistic limit.

With these ingredients, we see that (removing the $1/(2m)$ from the normalization of states so as to change from the relativistic normalization to the nonrelativistic normalization)

$$\frac{Qe}{2m}\,\bar{\mathcal{U}}(k,s)\Gamma^0(k'k)\mathcal{U}(k,s) \approx Qe\,F_1(q^2)\;\xi^\dagger(s)\xi(s) \tag{95}$$

while

$$\begin{aligned} \frac{Qe}{2m}\,\bar{\mathcal{U}}(k,s)\Gamma^k(k'k)\mathcal{U}(k,s) \;\; &\approx \;\; \frac{Qe}{2m}\,F_1(q^2)\;(k^k + k'^k)\xi^\dagger(s)\xi(s) \\ &\quad +i\,\frac{Qe}{2m}\,2[F_1(q^2) + \kappa F_2(q^2)]\epsilon^{ijk}q^j\,\xi^\dagger(s)\tfrac{1}{2}\sigma^i\xi(s). \end{aligned} \tag{96}$$

Taking $q^2 \to 0$ and comparing to our previous result for the interaction of nonrelativistic particles with the magnetic field, we see that we get the expected interactions with

$$g = 2(1+\kappa). \tag{97}$$

The first thing to notice is that $g = 2$ at zero order in perturbation theory. That's a remarkable prediction of the relativistic theory. This motivates us to calculate κ.

10 Calculation of κ

Exercise: Perform a calculation of κ at one loop order.

For the calculation of κ you will need the "Feynman parameter" representation of the denominators, which is based on the integral

$$\frac{1}{A_1 A_2 \cdots A_N} = \Gamma(N) \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_N \; \frac{\delta(1 - \sum \alpha_J)}{[\sum \alpha_J A_J]^N} \tag{98}$$

The proof of this is easier than the proof given in Peskin and Schroeder. You need just the integral that can be regarded as the definition of the gamma function:

$$\int_0^\infty \frac{dx}{x} \; x^N \; e^{-x} = \Gamma(N). \tag{99}$$

Using this, we have

$$\begin{aligned}
&\frac{1}{A_1 A_2 \cdots A_N} \\
&= \int_0^\infty d\beta_1 \cdots \int_0^\infty d\beta_N \; \exp(-\sum \beta_J A_J) \\
&= \int_0^\infty d\rho \int_0^\infty d\beta_1 \cdots \int_0^\infty d\beta_N \; \exp(-\sum \beta_J A_J)\, \delta(\rho - \sum \beta_J) \\
&= \int_0^\infty \frac{d\rho}{\rho} \int_0^\infty d\beta_1 \cdots \int_0^\infty d\beta_N \; \exp(-\sum \beta_J A_J)\, \delta(1 - \sum \beta_J/\rho) \\
&= \int_0^\infty \frac{d\rho}{\rho}\, \rho^N \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \; \exp(-\rho \sum \alpha_J A_J)\, \delta(1 - \sum \alpha_J) \\
&= \Gamma(N) \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \; \frac{1}{[\sum \alpha_J A_J)]^N} \; \delta(1 - \sum \alpha_J).
\end{aligned} \tag{100}$$

You will also need the integral

$$\int \frac{d^4 l}{(2\pi)^4} \; \frac{1}{[l^2 - \Lambda^2 + i\epsilon]^3} = -i(4\pi)^{-2} \; \frac{1}{2\,\Lambda^2}. \tag{101}$$

We will see how to prove this formula and its generalizations in the following sections.

The calculation is performed in some detail in Peskin and Schroeder.

The result is

$$\kappa = \frac{\alpha}{2\pi} \tag{102}$$

where

$$\alpha \equiv \frac{e^2}{4\pi} \tag{103}$$

is the “fine structure constant,” whose value is approximately 1/137.

Putting in the numerical value of α, we have

$$\tfrac{1}{2}\, g = 1 + \kappa = 1.00116(1) \tag{104}$$

where the “(1)” indicates a theoretical error estimate – the error on the last digit, which I guess might be $\pm 2(\alpha/\pi)^2 \approx 10^{-5}$. This result is confirmed by experiment for both electrons and muons.

To look further, you may want to consult the recent paper of Czarnecki and Marciano, hep-ph/0102122. The quantity called κ in these notes is called a in Czarnecki and Marciano. Its experimental value is

$$\kappa_{e^-} = 115\,965\,218.84(43) \times 10^{-11} \tag{105}$$

$$\kappa_{e^+} = 115\,965\,218.79(43) \times 10^{-11} \tag{106}$$

Comparison of this result with the theoretical prediction, including QED terms to α^4 and hadronic and electroweak loops, provides the best measurement of α. For the muon, an experiment at Brookhaven National Laboratory has just reported a result that is consistent with earlier results but has smaller errors by a factor of 3:

$$\kappa_\mu = 116\,592\,020(160) \times 10^{-11} \tag{107}$$

Note that we expect a slightly different result compared to the electron because the masses come into high order corrections. Also, note that the result is much less accurate than that for the electron. The best theoretical value is

$$\kappa_\mu = 116\,591\,597(67) \times 10^{-11} \tag{108}$$

These do not agree. It could be that the experimental systematic error is underestimated or that the theoretical error is underestimated. (The theoretical error comes almost entirely from the estimated error in the hadronic contributions, which are $6739(67) \times 10^{-11}$). It may also be that there is a contribution from new physical effects beyond the Standard Model. To understand this, note that if we had changed the mass in the denominator of our calculation from the muon mass ($m \approx 0.1$ GeV) to some new particle mass $M \approx 100$ GeV, we would get a contribution whose size can be roughly estimated as

$$\Delta\kappa \approx \frac{\alpha}{\pi}\,\frac{m^2}{M^2} \approx 2 \times 10^{-3} \times 10^{-6} = 200 \times 10^{-11} \tag{109}$$

This is the right size of effect to explain the discrepancy. Note that is important to be looking at the muon here. If we put the electron mass $m \approx 5 \times 10^{-4}$ GeV in the numerator the effect would be suppressed compared to the effect for the muon by a factor of $m_e^2/m_\mu^2 \approx 2.5 \times 10^{-5}$.