431/531 Class Notes 4

3.5 Frequency Domain Analysis (contd.)

Before we look at some more examples using our technique of complex impedance, let's look at some related general concepts.

3.5.1 Reactance

First, just a redefintion of what we already have learned. The term *reactance* is often used in place of impedance for capacitors and inductors. Reviewing our definitions of impedances from Section 3.2 we define the reactance of a capacitor $\tilde{X}_C$ to just be equal to its impedance: $\tilde{X}_C \equiv -\imath/(\omega C)$. Similarly, for an inductor $\tilde{X}_L \equiv \imath\omega L$. This is the notation used in the text.

However, an alternative but common useage is to define the reactances as real quantities. This is done simply by dropping the $\imath$ from the definitions above. The various reactances present in a circuit can by combined to form a single quantity X, which is then equal to the imaginary part of the impedance. So, for example a circuit with R, L, and C in series would have total impedance

$$\tilde{Z} = R + \imath X = R + \imath(X_L + X_C) = R + \imath(\omega L - \frac{1}{\omega C})$$

A circuit which is "reactive" is one for which X is non-negligible compared with R.

3.5.2 General Solution

As stated before, our technique involves solving for a single Fourier frequency component such as $\tilde{V} = Ve^{\imath(\omega t+\phi)}$. You may wonder how our results generalize to other frequencies and to input waveforms other than pure sine waves. The answer in words is that we Fourier decompose the input and then use these decomposition amplitudes to weight the output we found for a single frequency, V_{out}. We can formalize this within the context of the Fourier transform, whch will also allow us to see how our time-domain differential equation became transformed to an algebraic equation in frequency domain.

Consider the example of the RC low-pass filter, or integrator, circuit of Fig. 7. We obtained the differential equation given by Eq. 2. We wish to take the Fourier transform of this equation. Define the Fourier transform of $V(t)$ as

$$v(\omega) \equiv \mathcal{F}\{V(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{-\imath\omega t} V(t) \tag{10}$$

Recall that $\mathcal{F}\{dV/dt\} = \imath\omega\mathcal{F}\{V\}$. Therefore our differential equation becomes

$$\imath\omega v(\omega) + v(\omega)/(RC) = \mathcal{F}\{V_{\mathrm{in}}(t)\}/(RC) \tag{11}$$

Solving for $v(\omega)$ gives

$$v(\omega) = \frac{\mathcal{F}\{V_{\mathrm{in}}(t)\}}{1 + \imath\omega RC} \tag{12}$$

The general solution is then the real part of the inverse Fourier transform:

$$\tilde{V}(t) = \mathcal{F}^{-1}\{v(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega' e^{\imath\omega' t} v(\omega') \tag{13}$$

In the specific case we have considered so far of a single Fourier component of frequency ω, *i.e.* $\tilde{V}_{\text{in}} = V_i e^{\imath\omega t}$, then $\mathcal{F}\{\tilde{V}_{\text{in}}(t)\} = \sqrt{2\pi}\delta(\omega - \omega')$, and we recover our previous result for the transfer function:

$$\tilde{T} = \tilde{V}/\tilde{V}_{\text{in}} = \frac{1}{1 + \imath\omega RC} \tag{14}$$

For an arbitrary functional form for $V_{\text{in}}(t)$, one could use Eqns. 12 and 13. Note that one would go through the same steps if $V_{\text{in}}(t)$ were written as a Fourier series rather than a Fourier integral. Note also that the procedure carried out to give Eqn. 11 is formally equivalent to our use of the complex impedances: In both cases the differential equation is converted to an algebraic equation.

3.6 Phase Shift

We now need to discuss finding the phase ϕ of our solution. To do this, we proceed as previously, for example like the high-pass filter, but this time we preserve the phase information by not taking the modulus of $\tilde{V}_{\text{out}}$. The input to a circuit has the form $\tilde{V}_{\text{in}} = V_{\text{in}} e^{\imath(\omega t + \phi_1)}$, and the output $\tilde{V}_{\text{out}} = V_{\text{out}} e^{\imath(\omega t + \phi_2)}$. We are usually only interested in the phase difference $\phi_2 - \phi_1$ between input and output, so, for convenience, we can choose $\phi_1 = 0$ and set the *phase shift* to be $\phi_2 \equiv \phi$. Physically, we must choose the real or imaginary part of these expressions. Conventionally, the real part is used. So we have:

$$V_{\text{in}}(t) = \Re(\tilde{V}_{\text{in}}) = V_{\text{in}}(\omega)\cos(\omega t)$$

and

$$V_{\text{out}}(t) = \Re(\tilde{V}_{\text{out}}) = V_{\text{out}}(\omega)\cos(\omega t + \phi)$$

Let's return to our example of the high-pass filter to see how to calculate the phase shift. We rewrite the expression from Section 3.3 and then multiply numerator and denominator by the complex conjugate of the denominator:

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}} \left[\frac{R}{R - \imath/(\omega C)}\right] = V_{\text{in}} e^{\imath\omega t} \frac{1 + \imath/(\omega RC)}{1 + 1/(\omega RC)^2}$$

By recalling the general form $a + \imath b = \sqrt{a^2 + b^2}\, e^{\imath\phi}$, where $\phi = \tan^{-1}(b/a)$, we can write

$$1 + \imath/(\omega RC) = \left[1 + \left(\frac{1}{\omega RC}\right)^2\right]^{1/2} e^{\imath\phi}$$

allowing us to read off the phase shift:

$$\phi = \tan^{-1}(1/(\omega RC)) \tag{15}$$

Our solution for $\tilde{V}_{\text{out}}$ is then

$$\tilde{V}_{\text{out}} = \frac{V_{\text{in}} e^{\imath\omega t + \phi}}{\left[1 + (\frac{1}{\omega RC})^2\right]^{1/2}}$$

This, of course, yields the same $|\tilde{V}_{\rm out}|$ as we found before in Eqn. 6 of Section 3.3. But now we also have included the phase information. The "real" time-dependent solution is then just the real part of this:

$$V_{\rm out}(t) = \Re(\tilde{V}_{\rm out}) = V_{\rm out}\cos(\omega t + \phi)$$

where ϕ is given by Eqn. 15.

3.7 Power in Reactive Circuits

Recall that for DC voltages and currents the power associated with a circuit element carrying current I with voltage change V is just $P = VI$. Now, for time-varying voltages and currents we have to be more careful. We could still define an instantaneous power as the product $V(t)I(t)$. However, it is generally more useful to average the power over time.

3.7.1 General Result for AC

Since we are considering Fourier components, we will average the results over one period $T = 1/f = 2\pi/\omega$. Therefore, the time-averaged power is

$$<P>= \frac{1}{T}\int_0^T V(t)I(t)dt$$

where the brackets indicate the time average. Let the voltage and current be out of phase by an arbitrary phase angle ϕ. So we have $V(t) = V_0\cos(\omega t)$ and $I(t) = I_0\cos(\omega t + \phi)$. We can plug these into the expression for $<P>$ and simplify using the following: $\cos(\omega t + \phi) = \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)$; $\int_0^T \sin(\omega t)\cos(\omega t)dt = 0$; and $(1/T)\int_0^T \sin^2(\omega t)dt = (1/T)\int_0^T \cos^2(\omega t)dt = 1/2$. This yields

$$<P>= \frac{1}{2}V_0 I_0\cos\phi = V_{\rm RMS}I_{\rm RMS}\cos\phi \qquad (16)$$

In the latter expression we have used the "root mean squared", or *RMS*, amplitudes. Using voltage as an example, the RMS and standard amplitudes are related by

$$V_{\rm RMS} \equiv \left[\frac{1}{T}\int_0^T V^2(t)dt\right]^{1/2} = \left[\frac{1}{T}\int_0^T V_0^2\cos^2(\omega t)dt\right]^{1/2} = V_0/\sqrt{2} \qquad (17)$$

3.7.2 Power Using Complex Quantities

Our results above can be simply expressed in terms of $\tilde{V}$ and $\tilde{I}$. Equivalent to above, we start with $\tilde{V}(t) = V_0 e^{\imath\omega t}$ and $\tilde{I}(t) = I_0 e^{\imath(\omega t+\phi)}$. By noting that

$$\Re(\tilde{V}^*\tilde{I}) = \Re\left(V_0 I_0(\cos\phi + \imath\sin\phi)\right) = V_0 I_0\cos\phi$$

we identify an expression for average power which is equivalent to Eqn. 16 :

$$<P>= \frac{1}{2}\Re(\tilde{V}^*\tilde{I}) = \frac{1}{2}\Re(\tilde{V}\tilde{I}^*) \qquad (18)$$

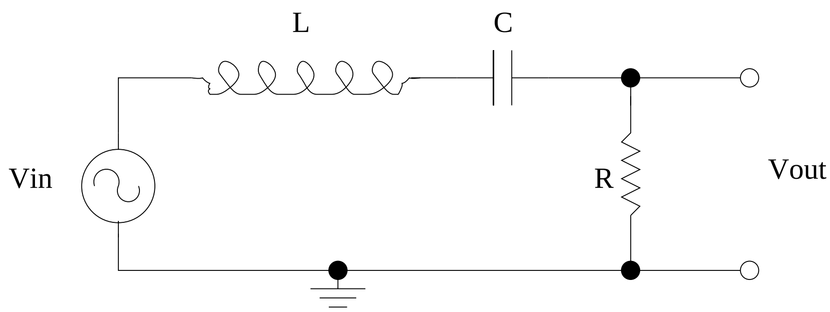

Figure 12: A RLC circuit. Several filter types are possible depending upon how $V_{\rm out}$ is chosen. In the case shown, the circuit gives a resonant output.

3.8 An RLC Circuit Example

We can apply our technique of impedance to increasingly more intricate examples, with no more effort than a commensurate increase in the amount of algebra. The RLC circuit of Fig. 12 exemplifies some new qualitative behavior.

We can again calculate the output using our generalized voltage divider result of Eqn. 5. In this case, the $\tilde{Z}_1$ consists of the inductor and capacitor in series, and $\tilde{Z}_2$ is simply R. So,

$$\tilde{Z}_1 = \imath\omega L - \imath/(\omega C) = \frac{\imath L}{\omega}\left(\omega^2 - \omega_0^2\right)$$

where we have defined the *LC resonant frequency* $\omega_0 \equiv 1/\sqrt{LC}$. We then obtain for the transfer function:

$$T(\omega) \equiv \frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \frac{R}{|R+\tilde{Z}_1|} = \frac{\omega\gamma}{[\omega^2\gamma^2 + (\omega^2-\omega_0^2)^2]^{1/2}}$$

where $\gamma \equiv R/L$ is the "R-L frequency".

$T(\omega)$ indeed exhibits a resonance at $\omega = \omega_0$. The quality factor Q, defined as the ratio of ω_0 to the width of the resonance is given by $Q \approx \omega_0/(2\gamma)$ for $\gamma \ll \omega_0$. Such circuits have many applications. For example, a high-Q circuit, where $V_{\rm in}(t)$ is the signal on an antenna, can be used as a receiver.

As was shown in class, we achieve different behavior if we choose to place the output across the capacitor or inductor, rather than across the resistor, as above. Rather than a resonant circuit, choosing $V_{\rm out} = V_C$ yields a low-pass filter of the form

$$T(\omega) = \frac{|-\imath/(\omega C)|}{|R+\tilde{Z}_1|} = \frac{\omega_0^2}{[\omega^2\gamma^2 + (\omega^2-\omega_0^2)^2]^{1/2}}$$

The cutoff frequency is ω_0, and for $\omega \gg \omega_0$ then $T \sim \omega^{-2}$ ("12 db per octave"), which more closely approaches ideal step function-like behavior than the RC low pass filter, for which $T \sim \omega^{-1}$ for $\omega \gg \omega_0$ ("6 db per octave"). As you might suspect, choosing $V_{\rm out} = V_L$ provides a high-pass filter with cutoff at ω_0 and $T \sim \omega^{-2}$ for $\omega \ll \omega_0$.

3.9 More Filters

3.9.1 Combining Filter Sections

Filter circuits can be combined to produce new filters with modified functionality. An example is the homework problem (6) of page 59 of the text, where a high-pass and a low-pass filter are combined to form a "band-pass" filter. As discussed at length in Section 1.5, it is important to design a "stiff" circuit, in which the next circuit element does not load the previous one, by requiring that the output impedance of the first be much smaller than the input impedance of the second. We can standardize this inequality by using a factor of 10 for the ratio $|\tilde{Z}_{\rm in}|/|\tilde{Z}_{\rm out}|$.

3.9.2 More Powerful Filters

This technique of cascading filter elements to produce a better filter is discussed in detail in Chapter 5 of the text. In general, the transfer functions of such filters take the form (for the low-pass case):

$$T(\omega) = \left[1 + \alpha_n (f/f_c)^{2n}\right]^{-1/2}$$

where f_c is the 3 db frequency, α_n is a coefficient depending upon the type of filter, and n is the filter "order," often equal to the number of filtering capacitors.

3.9.3 Active Filters

Filters involving LC circuits are very good, better than the simple RC filters, as discussed above. Unfortunately, inductors are, in practice, not ideal lumped circuit elements and are difficult to fabricate. In addition, filters made entirely from passive elements tend to have a lot of attenuation. For these reasons active filters are most commonly used where good filtering is required. These typically use operational amplifiers (which we will discuss later), which can be configured to behave like inductors, and can have provide arbitrary voltage gain. Again, this is discussed in some detail in Chapter 5. When we discuss op amps later, we will look at some examples of very simple active filters. At high frequencies (for example RF), op amps fail, and one most fall back on inductors.

4 Diode Circuits

The figure below is from Lab 2, which gives the circuit symbol for a diode and a drawing of a diode from the lab. Diodes are quite common and useful devices. One can think of a diode as a device which allows current to flow in only one direction. This is an over-simplification, but a good approximation.

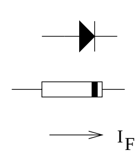

Figure 13: Symbol and drawing for diodes.

A diode is fabricated from a pn junction. Semi-conductors such as silicon or germanium can be "doped" with small concentrations of specific impurities to yield a material which conducts electricity via electron transport (n-type) or via holes (p-type). When these are brounght together to form a pn junction, electrons (holes) migrate away from the n-type (p-type) side, as shown in Fig. 14. This redistribution of charge gives rise to a potential gap ΔV across the junction, as depicted in the figure. This gap is $\Delta V \approx 0.7$ V for silicon and ≈ 0.3 V for germanium.

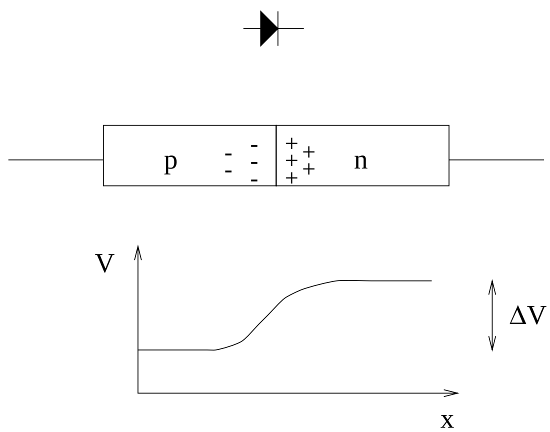

Figure 14: A pn junction, forming a voltage gap across the junction.

When a diode is now connected to an external voltage, this can effectively increase or decrease the potential gap. This gives rise to very different behavior, depending upon the polarity of this external voltage, as shown by the typical V-I plot of Fig. 15. When the diode is "reverse biased," as depicted in the figure, the gap increases, and very little current flows across the junction (until eventually at ~ 100 V field breakdown occurs). Conversely, a "forward biased" configuration decreases the gap, approaching zero for an external voltage equal to the gap, and current can flow easily. An analysis of the physics gives the form

$$I = I_S \left[e^{eV/kT} - 1 \right]$$

where I_S is a constant, V is the applied voltage, and $kT/e = 26$ mV at room temperature.

Thus, when reverse biased, the diode behaves much like an open switch; and when forward biased, for currents of about 10 mA or greater, the diode gives a nearly constant voltage drop of ≈ 0.6 V.

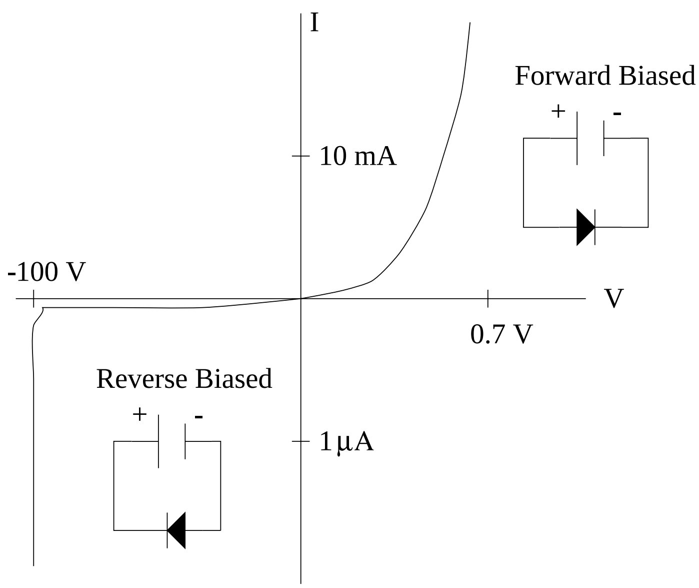

Figure 15: The V-I behavior of a diode.