431/531 Class Notes 3

3 Circuit Analysis in Frequency Domain

We now need to turn to the analysis of passive circuits (involving EMFs, resistors, capacitors, and inductors) in frequency domain. Using the technique of the complex impedance, we will be able to analyze time-dependent circuits algebraically, rather than by solving differential equations. We will start by reviewing complex algebra and setting some notational conventions. It will probably not be particularly useful to use the text for this discussion, and it could lead to more confusion. Skimming the text and noting results might be useful.

3.1 Complex Algebra and Notation

Let $\tilde{V}$ be the complex representation of V. Then we can write

$$\tilde{V} = \Re(\tilde{V}) + \imath\Im(\tilde{V}) = Ve^{\imath\theta} = V\left[\cos\theta + \imath\sin\theta\right]$$

where $\imath = \sqrt{-1}$. V is the (real) amplitude:

$$V = \sqrt{\tilde{V}\tilde{V}^*} = \left[\Re^2(\tilde{V}) + \Im^2(\tilde{V})\right]^{1/2}$$

where $*$ denotes complex conjugation. The operation of determining the amplitude of a complex quantity is called taking the *modulus*. The phase θ is

$$\theta = \tan^{-1}\left[\Im(\tilde{V})/\Re(\tilde{V})\right]$$

So for a numerical example, let a voltage have a real part of 5 volts and an imaginary part of 3 volts. Then $\tilde{V} = 5 + 3\imath = \sqrt{34}e^{\imath\tan^{-1}(3/5)}$.

Note that we write the amplitude of $\tilde{V}$, formed by taking its modulus, simply as V. It is often written $|\tilde{V}|$. We will also use this notation if there might be confusion in some context. Since the amplitude will in general be frequency dependent, it will also be written as $V(\omega)$. We will most often be interested in results expressed as amplitudes, although we will also look at the phase.

3.2 Ohm's Law Generalized

Our technique is essentially that of the Fourier transform, although we will not need to actually invoke that formalism. Therefore, we will analyze our circuits using a single Fourier frequency component, $\omega = 2\pi f$. This is perfectly general, of course, as we can add (or integrate) over frequencies if need be to recover a result in time domain. Let our complex Fourier components of voltage and current be written as $\tilde{V} = Ve^{\imath(\omega t+\phi_1)}$ and $\tilde{I} = Ie^{\imath(\omega t+\phi_2)}$.

Now, we wish to generalize Ohm's Law by replacing $V = IR$ by $\tilde{V} = \tilde{I}\tilde{Z}$, where $\tilde{Z}$ is the (complex) impedance of a circuit element. Let's see if this can work. We already know that a resistor R takes this form. What about capacitors and inductors?

Our expression for the current through a capacitor, $I = C(dV/dt)$ becomes

$$\tilde{I} = C\frac{d}{dt}Ve^{\imath(\omega t+\phi_1)} = \imath\omega C\tilde{V}$$

Thus, we have an expression of the form $\tilde{V} = \tilde{I}\tilde{Z}_C$ for the impedance of a capacitor, $\tilde{Z}_C$, if we make the identification $\tilde{Z}_C = 1/(\imath\omega C)$.

For an inductor of self-inductance L, the voltage *drop* across the inductor is given by Lenz's Law: $V = L(dI/dt)$. (Note that the voltage drop has the opposite sign of the induced EMF, which is usually how Lenz's Law is expressed.) Our complex generalization leads to

$$\tilde{V} = L\frac{d}{dt}\tilde{I} = L\frac{d}{dt}Ie^{\imath(\omega t+\phi_2)} = \imath\omega L\tilde{I}$$

So again the form of Ohm's Law is satisfied if we make the identification $\tilde{Z}_L = \imath\omega L$.

To summarize our results, Ohm's Law in the complex form $\tilde{V} = \tilde{I}\tilde{Z}$ can be used to analyze circuits which include resistors, capacitors, and inductors if we use the following:

- resistor of resistance R: $\tilde{Z}_R = R$
- capacitor of capacitance C: $\tilde{Z}_C = 1/(\imath\omega C) = -\imath/(\omega C)$
- inductor of self-inductance L: $\tilde{Z}_L = \imath\omega L$

3.2.1 Combining Impedances

It is significant to point out that because the algebraic form of Ohm's Law is preserved, impedances follow the same rules for combination in series and parallel as we obtained for resistors previously. So, for example, two capacitors in parallel would have an equivalent impedance given by $1/\tilde{Z}_p = 1/\tilde{Z}_1 + 1/\tilde{Z}_2$. Using our definition $\tilde{Z}_C = -\imath/\omega C$, we then recover the familiar expression $C_p = C_1 + C_2$. So we have for any two impedances in series (clearly generalizing to more than two):

$$\tilde{Z}_s = \tilde{Z}_1 + \tilde{Z}_2$$

And for two impedances in parallel:

$$\tilde{Z}_p = \left[1/\tilde{Z}_1 + 1/\tilde{Z}_2\right]^{-1} = \frac{\tilde{Z}_1\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}$$

And, accordingly, our result for a voltage divider generalizes (see Fig. 9) to

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}}\left[\frac{\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}\right] \tag{5}$$

Now we are ready to apply this technique to some examples.

3.3 A High-Pass RC Filter

The configuration we wish to analyze is shown in Fig. 10. Note that it is the same as Fig. 7 of the notes. However, this time we apply a voltage which is sinusoidal: $\tilde{V}_{\text{in}}(t) = V_{\text{in}}e^{\imath(\omega t+\phi)}$. As an example of another common variation in notation, the figure indicates that the input is sinusoidal ("AC") by using the symbol shown for the input. Note also that the input and output voltages are represented in the figure only by their amplitudes V_{in} and V_{out}, which also is common. This is fine, since the method we are using to analyze the circuit (complex impedances) shouldn't necessarily enter into how we describe the physical circuit.

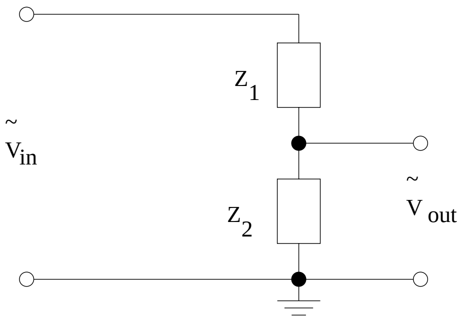

Figure 9: The voltage divider generalized.

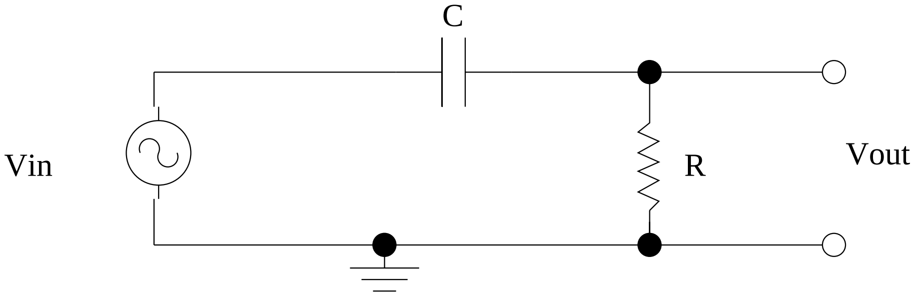

Figure 10: A high-pass filter.

We see that we have a generalized voltage divider of the form discussed in the previous section. Therefore, from Eqn. 5 we can write down the result if we substitute $\tilde{Z}_1 = \tilde{Z}_C = -\imath/(\omega C)$ and $\tilde{Z}_2 = \tilde{Z}_R = R$:

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}} \left[\frac{R}{R - \imath/(\omega C)} \right]$$

At this point our result is general, and includes both amplitude and phase information. Often, we are only interested in amplitudes. We can divide by $\tilde{V}_{\text{in}}$ on both sides and find the amplitude of this ratio (by multiplying by the complex conjugate then taking the square root). The result is often referred to as the *transfer function* of the circuit, which we can designate by $T(\omega)$.

$$T(\omega) \equiv \frac{|\tilde{V}_{\text{out}}|}{|\tilde{V}_{\text{in}}|} = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\omega RC}{[1 + (\omega RC)^2]^{1/2}} \tag{6}$$

Examine the behavior of this function. Its maximum value is one and minimum is zero. You should convince yourself that this circuit attenuates low frequencies and "passes" (transmits with little attenuation) high frequencies, hence the term *high-pass filter*. The cutoff between high and low frequencies is conventionally described as the frequency at which the transfer function is $1/\sqrt{2}$. This is approximately equal to an attenuation of 3 *decibels*, which is a description often used in engineering (see below). From Eqn. 6 we see that $T = 1/\sqrt{2}$ occurs at a frequency

$$2\pi f_{3\text{db}} = \omega_{3\text{db}} = 1/(RC) \tag{7}$$

The decibel scale works as follows: db$= 20\log_{10}(A_1/A_2)$, where A_1 and A_2 represent any real quantity, but usually are amplitudes. So a ratio of 10 corresponds to 20 db, a ratio of 2 corresponds to 6 db, $\sqrt{2}$ is approximately 3 db, *etc.*

3.4 A Low-Pass RC Filter

An analogy with the analysis above, we can analyze a low-pass filter, as shown in Fig. 11.

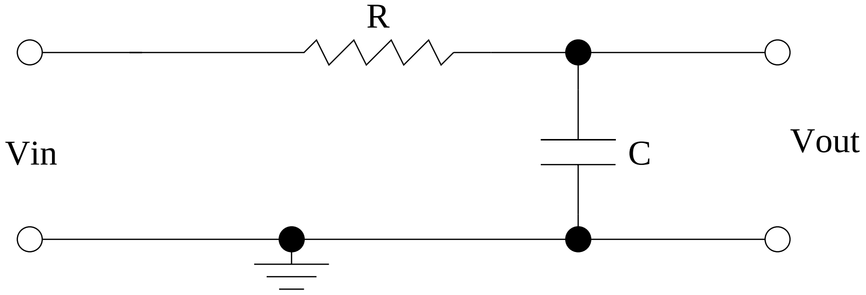

Figure 11: A low-pass filter.

You should find the following result for the transfer function:

$$T(\omega) \equiv \frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \frac{V_{\rm out}}{V_{\rm in}} = \frac{1}{[1+(\omega RC)^2]^{1/2}} \tag{8}$$

You should verify that this indeed exhibits "low pass" behavior. And that the 3 db frequency is the same as we found for the high-pass filter:

$$2\pi f_{\rm 3db} = \omega_{\rm 3db} = 1/(RC) \tag{9}$$

We note that the two circuits above are equivalent to the circuits we called "differentiator" and "integrator" in Section 2. However, the concept of high-pass and low-pass filters is much more general, as it does not rely on an approximation.

An aside. One can compare our results for the RC circuit using the complex impedance technique with what one would obtain by starting with the differential equation (in time) for an RC circuit we obtained in Section 2, taking the Fourier transform of that equation, then solving (algebraically) for the transform of $V_{\rm out}$. It should be the same as our result for the amplitude $V_{\rm out}$ using impedances. After all, that is what the impedance technique is doing: transforming our time-domain formuation to one in frequency domain, which, because of the possibility of analysis using a single Fourier frequency component, is particularly simple. This is discussed in more detail in the next notes.