431/531 Class Notes 2

1.5 Thevenin Theorem (contd.)

Recall that the Thevenin Theorem states that any collection of resistors and EMFs is equivalent to a circuit of the form shown within the box labelled "Circuit A" in the figure below. As before, the load resistor R_L is not part of the Thevenin circuit. The Thevenin idea, however, is most useful when one considers two circuits or circuit elements, with the first circuit's output providing the input for the second circuit. In Fig. 6, the output of the first circuit (A), consisitng of V_{TH} and R_{TH}, is fed to the second circuit element (B), which consists simply of a load resistance (R_L) to ground. This simple configuration represents, in a general way, a very broad range of analog electronics.

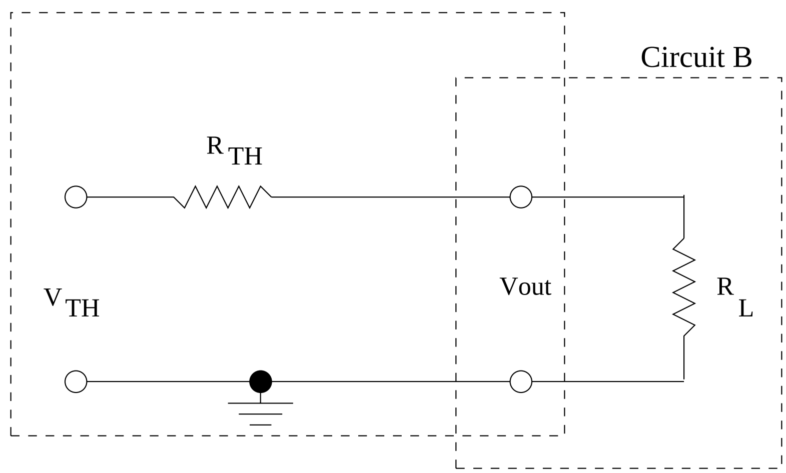

Figure 6: Two interacting circuits.

1.5.1 Avoiding Circuit Loading

V_{TH} is a voltage source. In the limit that $R_{\mathrm{TH}} \to 0$ the output voltage delivered to the load R_L remains at constant voltage. For finite R_{TH}, the output voltage is reduced from V_{TH} by an amount IR_{TH}, where I is the current of the complete circuit, which depends upon the value of the load resistance R_L: $I = V_{\mathrm{TH}}/(R_{\mathrm{TH}} + R_L)$.

Therefore, R_{TH} determines to what extent the output of the first circuit behaves as an ideal voltage source. An approximately ideal behavior turns out to be quite desirable in most cases, as V_{out} can be considered constant, independent of what load is connected. Since our combined equivalent circuit ($A + B$) forms a simple voltage divider, we can easily see what the requirement for R_{TH} can be found from the following:

$$V_{\mathrm{out}} = V_{\mathrm{TH}} \left[\frac{R_L}{R_{\mathrm{TH}} + R_L} \right] = \frac{V_{\mathrm{TH}}}{1 + (R_{\mathrm{TH}}/R_L)}$$

Thus, we should try to *keep the ratio R_{TH}/R_L small* in order to approximate ideal behavior and avoid "loading the circuit". A maximum ratio of 1/10 is often used as a design rule of thumb.

A good power supply will have a very small R_{TH}, typically much less than an ohm. For a battery this is referred to as its internal resistance. The dimming of one's car headlights when the starter is engaged is a measure of the internal resistance of the car battery.

1.5.2 Input and Output Impedance

Our simple example can also be used to illustrate the important concepts of input and output resistance. (Shortly, we will generalize our discussion and substitute the term "impedance" for resistance. We can get a head start by using the common terms "input impedance" and "output impedance" at this point.)

- The output impedance of circuit A is simply its Thevenin equivalent resistance R_{TH}. The output impedance is sometimes called "source impedance".

- The input impedance of circuit B is its resistance to ground from the circuit input. In this case it is simply R_L.

It is generally possible to reduce two complicated circuits, which are connected to each other as an input/output pair, to an equivalent circuit like our example. The input and output impedances can then be measured using the simple voltage divider equations.

2 RC Circuits in Time Domain

2.0.3 Capacitors

Capacitors typically consist of two electrodes separated by a non-conducting gap. The quantitiy capacitance C is related to the charge on the electrodes ($+Q$ on one and $-Q$ on the other) and the voltage difference across the capacitor by

$$C = Q/V_C$$

Capacitance is a purely geometric quantity. For example, for two planar parallel electrodes each of area A and separated by a vacuum gap d, the capacitance is (ignoring fringe fields) $C = \epsilon_0 A/d$, where ϵ_0 is the permittivity of vacuum. If a dielectric having dielectric constant κ is placed in the gap, then $\epsilon_0 \to \kappa\epsilon_0 \equiv \epsilon$. The SI unit of capacitance is the Farad. Typical laboratory capacitors range from $\sim$ 1pF to $\sim 1\mu$F.

For DC voltages, no current passes through a capacitor. It "blocks DC". When a time varying potential is applied, we can differentiate our defining expression above to get

$$I = C\frac{dV_C}{dt} \qquad (1)$$

for the current passing through the capacitor.

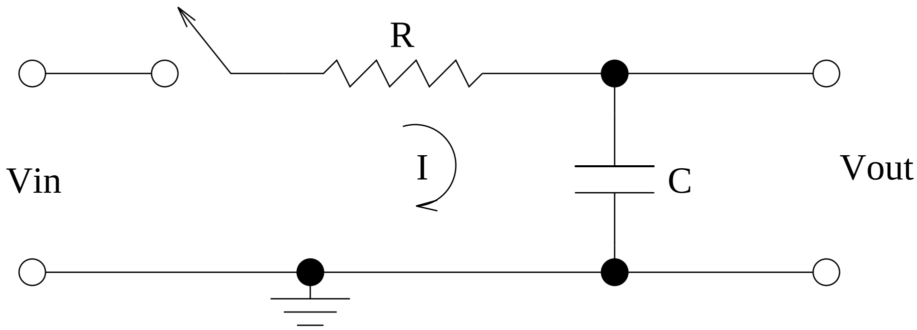

Figure 7: RC circuit — integrator.

2.0.4 A Basic RC Circuit

Consider the basic RC circuit in Fig. 7. We will start by assuming that $V_{\rm in}$ is a DC voltage source (*e.g.* a battery) and the time variation is introduced by the closing of a switch at time $t = 0$. We wish to solve for $V_{\rm out}$ as a function of time.

Applying Ohm's Law across R gives $V_{\rm in} - V_{\rm out} = IR$. The same current I passes through the capacitor according to $I = C(dV/dt)$. Substituting and rearranging gives (let $V \equiv V_C = V_{\rm out}$):

$$\frac{dV}{dt} + \frac{1}{RC}V = \frac{1}{RC}V_{\rm in} \tag{2}$$

The homogeneous solution is $V = Ae^{-t/RC}$, where A is a constant, and a particular solution is $V = V_{\rm in}$. The initial condition $V(0) = 0$ determines A, and we find the solution

$$V(t) = V_{\rm in}\left[1 - e^{-t/RC}\right] \tag{3}$$

This is the usual capacitor "charge up" solution.

Similarly, a capacitor with a voltage V_i across it which is discharged through a resistor to ground starting at $t = 0$ (for example by closing a switch) can in similar fashion be found to obey

$$V(t) = V_i e^{-t/RC}$$

2.0.5 The "RC Time"

In both cases above, the rate of charge/discharge is determined by the product RC which has the dimensions of time. This can be measured in the lab as the time during charge-up or discharge that the voltage comes to within $1/e$ of its asymptotic value. So in our charge-up example, Equation 3, this would correspond to the time required for $V_{\rm out}$ to rise from zero to 63% of $V_{\rm in}$.

2.0.6 RC Integrator

From Equation 2, we see that if $V_{\rm out} \ll V_{\rm in}$ then the solution to our RC circuit becomes

$$V_{\rm out} = \frac{1}{RC}\int V_{\rm in}(t)dt \tag{4}$$

Note that in this case $V_{\rm in}$ can be any function of time. Also note from our solution Eqn. 3 that the limit $V_{\rm out} \ll V_{\rm in}$ corresponds roughly to $t \ll RC$. Within this approximation, we see clearly from Eqn. 4 why the circuit above is sometimes called an "integrator".

2.0.7 RC Differentiator

Let's rearrange our RC circuit as shown in Fig. 8.

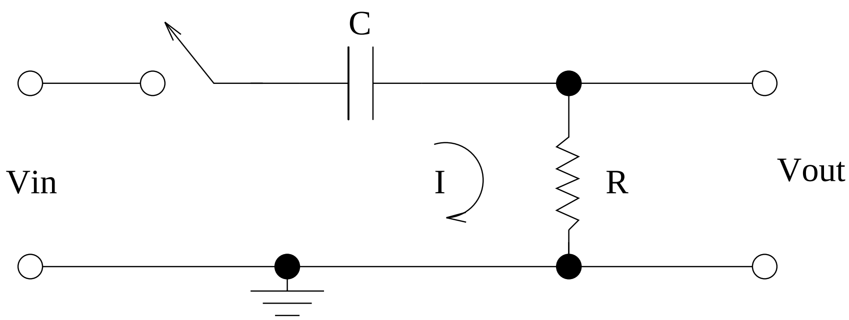

Figure 8: RC circuit — differentiator.

Applying Kirchoff's second Law, we have $V_{\rm in} = V_C + V_R$, where we identify $V_R = V_{\rm out}$. By Ohm's Law, $V_R = IR$, where $I = C(dV_C/dt)$ by Eqn. 1. Putting this together gives

$$V_{\rm out} = RC\frac{d}{dt}(V_{\rm in} - V_{\rm out})$$

In the limit $V_{\rm in} \gg V_{\rm out}$, we have a differentiator:

$$V_{\rm out} = RC\frac{dV_{\rm in}}{dt}$$

By a similar analysis to that of Section 2.0.6, we would see the limit of validity is the opposite of the integrator, *i.e.* $t \gg RC$.