Lecture 15 - Contour integration

§ Some basic preliminaries

- definitions of analytic and singular $f(z)$
- Cauchy-Goursat theorem: for $f(z)$ analytic on and within a closed contour C, then

$$\oint_C f(z)\,dz = 0$$

- Laurent series: for $f(z)$ analytic on and within the annular region between circular contours C_1 and C_2 centered on z_0, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\,(z - z_0)^n$$

- The quantity a_{-1} is the *residue* of f at z_0.

§ Residue theorem: For $f(z)$ analytic on and within C except for n singlular points within C with residues $\mathcal{R}_i$, then

$$\oint_C f(z)\,dz = 2\pi\imath \sum_{k=1}^{n} \mathcal{R}_k$$

§ Poles. If $\phi(z) = f(z)\,(z - z_0)^m$ is analytic at z_0, and $\phi(z_0) \neq 0$, then $f(z)$ has an m^{th} order *pole* at z_0. In this case:

$$f(z) = \sum_{n=-m}^{\infty} a_n\,(z - z_0)^n$$

$$\phi(z) = \sum_{n=-m}^{\infty} a_n\,(z - z_0)^{n+m}$$

Since $\phi(z)$ is analytic, a Taylor series for ϕ can be equated. Hence the residues are:

- for $m = 1$, $\mathcal{R} = \phi(z_0)$ (simple pole)
- for $m > 1$, $\mathcal{R} = \phi^{(m-1)}(z_0)/(m-1)!$

§ Jordan's Lemma. For (convergent) integrals of the form

$$\int_{-\infty}^{\infty} f(x)\, e^{\imath a x} dx$$

then for $a > 0$ a contour closed on the upper complex plane vanishes, while for $a < 0$, the contour should be closed on the lower plane.