Physics 410 practice problems for Final Exam
Fall 2007, Frey

Closed book exam. No calculators.

1. Consider the Sturm-Liouville equation $\mathcal{L}u_i(x) + \lambda_i w(x) u_i(x) = 0$. $\mathcal{L}$ is Hermitian over the interval $[a, b]$ if

$$\int_a^b u_i^* \mathcal{L} u_j dx = \int_a^b u_j (\mathcal{L} u_i)^* dx$$

   (a) Express the above equation in Dirac notation.

   (b) Use the integral relation above and the S-L equation to show that (i) the eigenvalues are real, and (ii) the eigenfunctions are orthogonal.

2. Consider the eigenvalue equation $\mathsf{A}\,|v\rangle = \lambda\,|v\rangle$, where $\mathsf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(a) Find the eigenvalues.

(b) Find the (normalized) eigenvectors.

(c) Let $\mathsf{A}' = \mathsf{SAS}^{-1}$ be the diagonalized matrix. Write down S.

3. (a) Represent the following function with a Fourier series:

$$f(x) = \begin{cases} -h/2 & , -L < x < 0 \\ +h/2 & , 0 < x < L \end{cases}$$

where h is a constant.

(b) Indicate *qualitatively* what would happen to your representation of $f(x)$ if the series were truncated so that, say, only the first ~ 100 terms are kept.

4. Let $f(x)$ be expressed in a Hilbert space of Legendre function:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

over the x-interval $[-1, 1]$. Show that

$$\int_{-1}^{+1} [f(x)]^2 \, dx = 2 \sum_{n=0}^{\infty} \frac{a_n^2}{2n+1} P_n(x)$$

5. Using contour integration, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$$

6. Show that if matrices A and B are diagonalized by the *same* unitary similarity transformation, then $[\mathsf{A}, \mathsf{B}] = 0$.

7. Consider the equation for a driven, damped oscillator:

$$\frac{d^2x}{dx^2} + 2\alpha\frac{dx}{dt} + \omega_0^2\, x(t) = f(t)$$

where α and ω_0 are constants. Let $X(\omega)$ be the Fourier transform of $x(t)$ and $F(\omega)$ the Fourier transform of $f(t)$.

(a) Find an expression for $X(\omega)$.

(b) Treating ω as complex, does $X(\omega)$ have any singular points? If so, where? (Assume that $F(\omega)$ is analytic.)

(c) Write down the expression for $x(t) = \mathcal{F}^{-1}\{X(\omega)\}$.

(d) Now, let $f(t) = a$, $0 < t < \tau$, where a is a constant; otherwise $f(t) = 0$. Find $F(\omega)$ and solve for $x(t)$ using contour integration.

8. Consider $\mathcal{L}$ of the form

$$\mathcal{L} = p(x)\frac{d^2}{dx^2} + \frac{dp(x)}{dx}\frac{d}{dx} + q(x)$$

Assuming p and q real, but $u(x)$ and $v(x)$ complex, show that

$$\int_a^b v^* \mathcal{L} u dx - \int_a^b u \mathcal{L} v^* dx = [pv^* u' - puv'^*]_a^b$$

Possibly Useful Relations

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{\imath kx}\, dx$$

$$\mathcal{F}^{-1}\{F(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{-\imath kx}\, dk$$

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\imath k(x-x')}\, dk$$

$$\int_{-1}^{+1} [P_n]^2\, dx = \frac{2}{2n+1}$$

$$\int_{0}^{2\pi} [sin(nx)]^2\, dx = \pi; , \quad n \neq 0$$

$$\int_{0}^{2\pi} [cos(nx)]^2\, dx = \pi\ , \quad n \neq 0$$