Physics 410 HW#4 Solutions

10.1.8, 10.1.13, 10.1.16, 10.1.17, 10.2.5, 10.2.6

10.1.8 [Note: The problem assumes $\mathcal{L}$, y_1, and y_2 are real. Otherwise, the LHS of the given equation is equivalent to our definition for $\mathcal{L}$ being Hermitian:

$$\int_a^b y_2^* \mathcal{L} y_1\,dx = \int_a^b y_1 (\mathcal{L} y_2)^*\,dx.$$

Here, we will proceed with $\mathcal{L}$ assumed real, but y_1, y_2 complex.]

$\mathcal{L}$ is self-adjoint: $\mathcal{L}y = \frac{d}{dx}\left[p\frac{dy}{dx}\right] + qy$.

Now,

$$\int_a^b y_2^* \mathcal{L} y_1\,dx = \underbrace{\int_a^b y_2^* \frac{d}{dx}\left[p\frac{dy_1}{dx}\right]dx}_{\equiv I_1} + \int_a^b y_2^* q y_1\,dx \quad (1a)$$

Integrate I_1 by parts: $\int f\,dg = fg - \int g\,df$, $f = y_2^*$, $g = py_1'$

$$\therefore I_1 = \left[y_2^* p y_1'\right]_a^b - \int_a^b p y_1' y_2^{*\prime}\,dx. \quad (1b)$$

Similarly,

$$\int_a^b y_1 \mathcal{L} y_2^*\,dx = \int_a^b y_1 \frac{d}{dx}\left[p\frac{dy_2^*}{dx}\right]dx + \int_a^b y_1 q y_2^*\,dx$$

$$= \left[y_1 p y_2^{*\prime}\right]_a^b - \int_a^b y_1' p y_2^{*\prime}\,dx + \int_a^b y_1 q y_2^*\,dx \quad (2)$$

We see that the RHS of Eq. (1) is the same as Eq. (2) except for the integrated terms.

So,

$$\int_a^b y_2^* \mathcal{L} y_1 \, dx - \int_a^b y_1 \mathcal{L} y_2^* \, dx = \left[p y_1' y_2^* \right]_a^b - \left[y_1 p y_2^{*\prime} \right]_a^b .$$

If we let y_1 and y_2 be real and combine the integrated terms, we get the required result:

$$\int_a^b [y_2 \mathcal{L} y_1 - y_1 \mathcal{L} y_2] \, dx = \left[p(y_1' y_2 - y_1 y_2') \right]_a^b \quad //$$

$\boxed{10.1.13}$ Use our def. of adjoint operator (Eq 10.27):

(1) $\int \psi_1^* A^\dagger \psi_2 \, dx = \int \psi_2 (A \psi_1)^* dx$ [or $\langle \psi_1 | A^\dagger | \psi_2 \rangle = \langle \psi_2 | A | \psi_1 \rangle^\dagger$

In the following, we use these definitions: $A = UV$, $\varphi_1 = V\psi_1$, $\varphi_2 = U^\dagger \psi_2$

So, $\int \psi_1^* (UV)^\dagger \psi_2 \, dx = \int \psi_1^* A^\dagger \psi_2 = \int \psi_2 (A\psi_1)^*$, by (1)

$$= \int \psi_2 (UV\psi_1)^* = \int \psi_2 U^* \varphi_1^* = \int \psi_2 (U\varphi_1)^*$$

$$= \int \varphi_1^* U^\dagger \psi_2 \text{ , by (1)}$$

$$= \int U^\dagger \psi_2 \, \varphi_1^* = \int \varphi_2 (V\psi_1)^*$$

$$= \int \psi_1^* V^\dagger \varphi_2 \text{ , by (1)}$$

$$= \int \psi_1^* V^\dagger U^\dagger \psi_2 \, dx$$

$\therefore (UV)^\dagger = V^\dagger U^\dagger$ //

$\boxed{10.1.16}$ $\mathcal{L}$ Hermitian. Show $\langle \mathcal{L}^2 \rangle \geq 0$

$$\langle \mathcal{L} \rangle \equiv \int \psi^* \mathcal{L} \psi \, d\tau \;; \quad \int \psi_1^* \mathcal{L} \psi_2 \, d\tau = \int \psi_2 (\mathcal{L}\psi_1)^* d\tau \text{ for } \mathcal{L} \text{ Herm.}$$

$$\rightarrow \langle \mathcal{L}^2 \rangle = \int \psi^* \mathcal{L}^2 \psi \, d\tau = \int \psi^* \mathcal{L} (\mathcal{L}\psi) \, d\tau \equiv \int \psi^* \mathcal{L} \varphi \, d\tau$$

$$= \int \varphi (\mathcal{L}\psi)^* d\tau \,, \text{ since } \mathcal{L} \text{ Hermitian}$$

(eqn. 9.26 of text)

$$= \int (\mathcal{L}\psi)(\mathcal{L}\psi)^* d\tau$$

$$= \int \varphi \varphi^* \, d\tau$$

$$= \int |\varphi|^2 d\tau \geq 0 \quad \text{for any } \varphi \,. \quad //$$

$\boxed{10.1.17}$ $\langle A \rangle = \int \psi^* A \psi \, dx$

Show that $\langle A \rangle$ real $\Rightarrow$ A Hermitian.

$$\langle A \rangle^* = \left(\int \psi^* A \psi \, dx \right)^* = \int \psi A^* \psi^* \, dx$$

$$= \int \psi (A\psi)^* \, dx$$

$$\therefore \langle A \rangle = \langle A \rangle^* \Rightarrow \int \psi^* A \psi \, dx = \int \psi (A\psi)^* \, dx$$

$\Rightarrow$ A Hermitian with respect to ψ. //

$\boxed{10.2.5}$ Let $u_n(x) \equiv P_n'(x) = \frac{dP_n}{dx}$ (4)

a) Legendre eqn.: $(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$

or (1) $\mathcal{L}_L P_n = -\lambda_n P_n$ in Sturm-Liouville form,

with $\mathcal{L}_L = \frac{d}{dx}\left[p\frac{d}{dx}\right]$, $p = (1-x^2)$, $\lambda_n = n(n+1)$, n integer.

To get Eqn. for u_n, take $\frac{d}{dx}$ of Eq (1):

$$-n(n+1)P_n' = (1-x^2)P_n''' - 2xP_n'' - 2xP_n'' - 2P_n'$$

or $\underbrace{(1-x^2)u_n'' - 4xu_n'}_{\mathcal{L}_1 u_n} = -[n(n+1)-2]u_n$

So we get an e-value problem for $u_n = P_n'$, but $\mathcal{L}_1$ is not self-adjoint. So we make it so:

$\mathcal{L}_2 = r(x)\mathcal{L}_1 = \left\{\frac{1}{p_0}\exp\left[\int^x \frac{p_1}{p_0}dx'\right]\right\}\mathcal{L}_1$ will be self-adjoint.

$$\int^x \frac{p_1}{p_0}dx' = \int^x \frac{-4x'}{1-x'^2}dx' = 2\int^{1-x^2}\frac{dt}{t}, \quad \text{where } t = 1-x^2$$

$$= 2\ln(1-x^2) = \ln\left((1-x^2)^2\right)$$

$$\Rightarrow r(x) = \frac{(1-x^2)^2}{1-x^2} = 1-x^2$$

$$\therefore \mathcal{L}_2 u_n = r(x)\mathcal{L}_1 u_n = r(x)\{-[n(n+1)-2]\}u_n$$

or: $\mathcal{L}_2 u_n = -\lambda_n w(x)\, u_n$

$$\Rightarrow (1-x^2)^2 u_n'' - 4x(1-x^2)u_n' = -[n(n+1)-2](1-x^2)\, u_n$$

//

$\mathcal{L}_2$ is now self-adjoint with $p = (1-x^2)^2$.

[Check B.C.'s: $\left[p\,(u_n' u_m - u_n u_m')\right]_{-1}^{+1} = 0$ since $p(\pm 1) = 0$.

$\therefore$ $\mathcal{L}_2$ is Hermitian and we expect

(1) $\lambda_n = -[n(n+1)-2]$ real

(2) completeness of the u_n's

(3) orthogonality: $\int_{-1}^{1} u_n u_m w\, dx = C_n \delta_{n,m}$.]

(b) Since $w = 1-x^2$, as noted above, the fact that $\mathcal{L}_2$ is Hermitian guarantees $\int_{-1}^{1} P_n'(x) P_m'(x)(1-x^2)\, dx = C_n \delta_{n,m}$.

We can also show this explicitly: $\int f\, dg = fg - \int g\, df$

$$\int_{-1}^{1} \underbrace{P_n'(x)}_{dg} \underbrace{P_m'(x)(1-x^2)}_{f} dx = P_m'(x)P_n'(x)(1-x^2)\Big|_{-1}^{1} - \int_{-1}^{1} P_n(x)\frac{d}{dx}\left[P_m'(x)(1-x^2)\right]dx$$

$$= 0 - \int_{-1}^{1} P_n(x)\, \mathcal{L}_L P_m(x)\, dx$$

$$= 0 - \int_{-1}^{1} P_n(x)[-n(n+1)]P_m(x)\, dx$$

$$= n(n+1)\,\delta_{n,m} \quad \text{since } \int_{-1}^{1} P_n P_m\, dx = \delta_{n,m}$$

//

$\boxed{10.2.6}$ Given $\{u_n\}$ such that

(1) $\mathcal{L} u_n = \frac{d}{dx}\left[p(x)\frac{du_n}{dx}\right] + \lambda_n w(x) u_n = 0$

and (2) $\int_a^b u_n u_m w \, dx = 0$, $\lambda_m \neq \lambda_n$.

multiply Eq. (1) by u_m^* and integrate :

$$0 = \int_a^b u_m^* \mathcal{L} u_n \, dx = \underbrace{\int_a^b u_m^* \frac{d}{dx}\left[p(x)\frac{du_n}{dx}\right] dx}_{\text{integrate by parts}} + \lambda_n \underbrace{\int_a^b u_m^* u_n w \, dx}_{=0 \text{ by Eq (2)}}$$

$$0 = \int_a^b u_m^* \frac{d}{dx}\left[p(x)\frac{du_n}{dx}\right] dx$$

$\int_a^b f\,dg = fg\Big|_a^b - \int_a^b g\,df$

$f = u_m^*$, $g = p(x)\frac{du_n}{dx}$

$$= u_m^* p \frac{du_n}{dx}\Bigg|_a^b - \int_a^b \frac{du_m^*}{dx}\frac{du_n}{dx} p(x)\, dx$$

$\therefore$ with appropriate boundary conditions,

$$u_m^* p u_n' \Big|_a^b = 0 \, ,$$

then $\int_a^b u_m^{*\prime} u_n' \, p(x) \, dx = 0$, as required. //