Physics 410 Hw#3 Solutions
3.5.9, 3.6.20; 1.15.1, 1.15.6, 1.15.9;
9.7.6, 10.1.1

$\boxed{3.5.9}$ $\underset{\sim}{A}$ and $\underset{\sim}{B}$ have same eigenvalues ($\underset{\sim}{A}^\dagger = \underset{\sim}{A}$, $\underset{\sim}{B}^\dagger = \underset{\sim}{B}$)

$\Rightarrow$ $\underset{\sim}{A} = \underset{\sim}{U}\,\underset{\sim}{B}\,\underset{\sim}{U}^\dagger$, $\underset{\sim}{U}$ unitary.

$\exists$ $\tilde{U}_1 \ni$ $\underset{\sim}{A}' = \underset{\sim}{U}_1 \underset{\sim}{A}\, \underset{\sim}{U}_1^\dagger$ is diagonal w/ eigenvalues as diagonal elements,

and $\tilde{U}_2 \ni$ $\underset{\sim}{B}' = \underset{\sim}{U}_2 \underset{\sim}{B}\, \underset{\sim}{U}_2^\dagger$ " " " " " ,

where $\tilde{U}_1$ and $\tilde{U}_2$ are unitary.

$$\therefore \quad \underset{\sim}{A}' = \underset{\sim}{B}'$$

$$U_1 A U_1^\dagger = U_2 B U_2^\dagger .$$

multiply on left by $U_1^\dagger$ and right by U_1:

$$\hookrightarrow \quad \underbrace{U_1^\dagger U_1 A U_1^\dagger U_1}_{=A} = U_1^\dagger U_2 B U_2^\dagger U_1$$

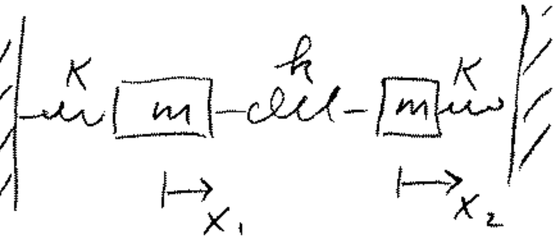
A, since $U_1^\dagger U_1 = U_1 U_1^\dagger = \underset{\sim}{1}$.

Let $U_3 = U_1^\dagger U_2$, where U_3 is unitary since

$$U_3 U_3^\dagger = (U_1^\dagger U_2)(U_1^\dagger U_2)^\dagger = U_1^\dagger U_2 U_2^\dagger U_1^{\dagger\dagger} = U_1^\dagger \underset{=}{1} U_1 = \underset{=}{1} .$$

$\therefore$ $\underset{\sim}{A} = \underset{\sim}{U}_3 \underset{\sim}{B}\, \underset{\sim}{U}_3^\dagger$ with U_3 unitary. //

$\boxed{3.6.20}$ (We set this up in class.)



(a) left mass:

$$ma_1 = \sum F_x \Rightarrow m\ddot{x}_1 = -Kx_1 + k\Delta x$$
$$= -(K+k)x_1 + kx_2$$

$x_2 = x_1 + \Delta x$, $\Delta x = x_2 - x_1$

right: $m\ddot{x}_2 = -Kx_2 - k\Delta x = kx_1 - (K+k)x_2$.

Look for common-frequency solutions (normal modes):

$x_i = X_{0i}\, e^{i\omega t}$, $i = 1, 2$. Substituting into above:

$$\begin{cases} -m\omega^2 x_1 = -(K+k)x_1 + kx_2 \\ -m\omega^2 x_2 = kx_1 - (K+k)x_2 \end{cases}$$

or: $\underset{\sim}{A}|v\rangle = \omega^2|v\rangle$ (eigenvalue eqn.)

with $\underset{\sim}{A} = \frac{1}{m}\begin{pmatrix} K+k & -k \\ -k & K+k \end{pmatrix}$, $|v\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

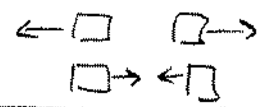
(b) secular: $0 = |\underset{\sim}{A} - \omega^2 \underset{\sim}{1}| = \begin{vmatrix} \omega_\alpha^2 - \omega^2 & \omega_\beta^2 \\ \omega_\beta^2 & \omega_\alpha^2 - \omega^2 \end{vmatrix} = (\omega_\alpha^2 - \omega^2)^2 - \omega_\beta^2$,

where $\omega_\alpha^2 \equiv \frac{K+k}{m}$, $\omega_\beta^2 \equiv k/m$. Quadratic eqn. for ω^2

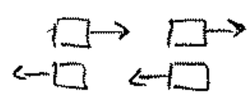
$\rightarrow \omega^2 = \omega_\alpha^2 \pm \omega_\beta^2 \Rightarrow \left\{ \begin{array}{l} \omega_1^2 = \omega_\alpha^2 + \omega_\beta^2 = (K+2k)/m \\ \omega_2^2 = \omega_\alpha^2 - \omega_\beta^2 = K/m \end{array} \right\}$ eigenvalues

(c) e-vectors: $0 = (\underset{\sim}{A} - \omega_1^2 \underset{\sim}{1})|v_1\rangle = (-k/m)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

mode 1 $\left\{ \Rightarrow |v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right.$ and $x_1(t) = -x_2(t)$ with freq. $f_1 = \omega_1/2\pi$.



mode 2 $\left\{ \begin{array}{l} 0 = (\underset{\sim}{A} - \omega^2 \underset{\sim}{1})|v_2\rangle = (-k/m)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \Rightarrow |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } x_1(t) = x_2(t) \text{ with freq. } f_2 = \omega_2/2\pi \end{array} \right.$



$\boxed{1.15.1}$

$$\delta_n(x) = \begin{cases} 0, & x < -1/2n \\ n, & -1/2n < x < 1/2n \\ 0, & x > 1/2n \end{cases}$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = \lim_{n\to\infty} n \int_{-1/2n}^{1/2n} f(x)\, dx$$

Now, $f(x) = f(0) + x f'(0) + \frac{1}{2}x^2 f''(0) + \cdots \cong f(0) + x f'(0) + \cdots$
for $x \to 0$

$$\therefore \lim_{n\to\infty} n \int_{-1/2n}^{1/2n} f(x)\, dx = \lim_{n\to\infty} \Big\{ n f(0)\left[\tfrac{1}{2n} + \tfrac{1}{2n}\right] + n f'(0) \tfrac{1}{2}\left[\left(\tfrac{1}{2n}\right)^2 - \left(\tfrac{1}{2n}\right)^2\right] + K_1 n \left(\tfrac{1}{n}\right)^3 + K_2 \tfrac{1}{n^4} + \cdots$$

$$= f(0) + \lim_{n\to\infty} \Big\{ K_1 \tfrac{1}{n^2} + K_2 \tfrac{1}{n^4} + \cdots \Big\}$$

$$= f(0)$$

$$\therefore \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0) \quad //$$

$\boxed{1.15.6}$ $I \equiv \int_{-\infty}^{\infty} \delta(a(x-x_1))\, f(x)\, dx$ Let $y = a(x-x_1)$, $x = \frac{y}{a} + x_1$
$dy = a\,dx$

$$\therefore I = \frac{1}{a} \int_{-\infty}^{\infty} \delta(y) f\left(\tfrac{y}{a} + x_1\right) dy = \frac{1}{a} f(x_1)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} \delta(x - x_1)\, f(x)\, dx .$$

$\therefore$ we can write $\delta[a(x-x_1)] = \frac{1}{a}\, \delta(x - x_1) \quad //$

[It is noted, but we are not asked to show, that the same result holds for $a = -|a|$. So we can write:

$$\delta[a(x-x_1)] = \frac{1}{|a|}\, \delta(x - x_1) \quad \text{for } a \text{ pos. or neg.}$$]

$\boxed{1.15.9}$ Show $\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = -f'(0)$

Use integration by parts: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Let $u = f(x)$, $v = \delta(x) \Rightarrow du = f'(x)\,dx$, $dv = \delta'(x)\,dx$.

Then $\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = \cancel{\delta(x) f(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x) f'(x)\, dx$

0

We need only assume that $f(x)$ is continuous at $x = 0$ so that $f'(0)$ exists. Then

$$-\int_{-\infty}^{\infty} \delta(x) f'(x)\, dx = -f'(0) \quad . //$$

$\boxed{9.7.6}$ $\varphi = \frac{Z}{4\pi\varepsilon_0} \frac{e^{-ar}}{r}$. Find $\rho = -\varepsilon_0 \nabla^2 \varphi$.

Several ways to do this. We need to be careful because of the singular behavior at $r \to 0$. We will use the δ-function to represent this.

① We can simply use $\nabla^2 f$ in spherical coordinates:
From ch. 1, inside front cover of text, math hand book, etc:

$$\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r) \quad \text{since } \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \varphi} = 0.$$

and eqn. 1.170: $\nabla^2 \left(\frac{1}{r}\right) = -4\pi\, \delta(\vec{r})$

↓

② We try a more physical approach.

The $\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r})$ corresponds to

$$\nabla^2\varphi = -\rho/\varepsilon_0 \quad \text{for a point charge at } \vec{r}=0$$

where $\varphi = \frac{Z}{4\pi\varepsilon_0}\frac{1}{r}$ and $\rho = Z\,\delta(\vec{r})$.

In terms of the electric field, this corresponds to

$$\vec{E} = \frac{Z}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}, \quad \text{where } \vec{E} = -\vec{\nabla}\varphi.$$

Then, Gauss's Law is $\vec{\nabla}\cdot\vec{E} = \frac{Z}{4\pi\varepsilon_0}\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \rho/\varepsilon_0 = \frac{Z}{\varepsilon_0}\delta(\vec{r})$

$$\therefore\ \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta(\vec{r}).$$

So, $\nabla^2\varphi = \frac{Z}{4\pi\varepsilon_0}\vec{\nabla}\cdot\hat{r}\frac{d}{dr}\left(\frac{e^{-ar}}{r}\right) = \left(\frac{Z}{4\pi\varepsilon_0}\right)\vec{\nabla}\cdot\hat{r}\left(-\frac{1}{r^2}-\frac{a}{r}\right)e^{-ar}$

$$= -\left(\frac{Z}{4\pi\varepsilon_0}\right)\vec{\nabla}\cdot\hat{r}\,\frac{e^{-ar}}{r^2}(1+ar)$$

$$= -\left(\frac{Z}{4\pi\varepsilon_0}\right)\left\{e^{-ar}(1+ar)\underbrace{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)}_{=4\pi\delta(\vec{r})} - \frac{\hat{r}}{r^2}\cdot\vec{\nabla}\left(e^{-ar}(1+ar)\right)\right\}$$

Now, $e^{-ar}(1+ar)\,\delta(\vec{r}) \rightarrow e^{0}(1+0)\,\delta(\vec{r}) = \delta(\vec{r})$, since the product is zero except at $r=0$.

Finally, $\hat{r}\cdot\vec{\nabla} = \frac{\partial}{\partial r}$, and $\frac{d}{dr}\left(e^{-ar}(1+ar)\right) = \left(-a(1+ar)+a\right)e^{-ar}$

$$= -a^2 r\, e^{-ar}$$

$$\therefore\ \rho = -\varepsilon_0\nabla^2\varphi = \varepsilon_0\left(\frac{Z}{4\pi\varepsilon_0}\right)\left\{4\pi\delta(\vec{r}) - \frac{a^2}{r}e^{-ar}\right\}$$

$$= Z\,\delta(\vec{r}) - \frac{Za^2}{4\pi}\frac{e^{-ar}}{r} \quad //$$

A pt. charge Z at $\vec{r}=0$ + a sph. symmetric dist. of opposite sign.

$\boxed{9.1.1}$ We worked this out in class ....

Laguerre eqn.: $0 = xy'' + (1-x)y' + ay$
$= \mathcal{L}_{Lag.} + ay$

General form : $0 = p_0 y'' + p_1 y' + p_2 y + \lambda w y$
$= \mathcal{L}_{gen} + \lambda w y$

Sturm-Liouville (self-adjoint) form : $0 = py'' + p'y + qy + \lambda w y$
$= \mathcal{L}_{SL} + \lambda w y$

Self adjoint $\Rightarrow$ $p_1 = p' = p_0'$

For Laguerre, $\left.\begin{array}{l} p_0 = x \\ p_1 = (1-x) \end{array}\right\} p_0' = 1 \neq p_1$ $\therefore$ <u>NOT</u> self-adjoint.

As disussed in class, we can <u>make</u> it self adjoint by multiplying by $\frac{f}{p_0}$ where $f = \exp\left[\int^x \frac{p_1}{p_0} dx'\right]$.

So, $f = \exp\left[\int^x \frac{1-x'}{x'} dx'\right] = e^{\ln x} e^{-x} = x e^{-x}$ for Laguerre,

$\rightarrow 0 = xe^{-x} y'' + (1-x)e^{-x} y' + ae^{-x} y$ //

Check : $p_0' = \frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x} = (1-x)e^{-x} = p_1$ ✓

$\therefore$ it is now self-adjoint, with

$p = xe^{-x}$
$\lambda = a$
$w = e^{-x}$
$q = 0$

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