Physics 410 HW#1 Solutions

3.1.1, 3.1.2, 3.2.3, 3.2.20, 3.2.32, 3.2.34, 3.2.36, 3.3.9

$\boxed{3.1.1}$ (a) $\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$|\underset{\sim}{A}| = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (-1) = -1 \ .//$$

(b) $\underset{\sim}{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

$$|\underset{\sim}{A}| = 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 2 \cdot (-1) \cdot \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= -5 + (-2) \cdot 3 = -11 \ .//$$

(c) $\underset{\sim}{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$

$$|\underset{\sim}{A}| = \frac{1}{\sqrt{2}} \cdot \left\{ \sqrt{3} \cdot (-1)^{3+4} \cdot \begin{vmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 2 & \sqrt{3} \end{vmatrix} \right\}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \cdot (-1) \cdot \sqrt{3} \cdot (-1)^{2+1} \cdot \begin{vmatrix} \sqrt{3} & 0 \\ 2 & \sqrt{3} \end{vmatrix}$$

$$= \frac{3}{\sqrt{2}} \cdot 3 = 9/\sqrt{2} \ .//$$

$\boxed{3.1.2}$ $\begin{pmatrix} 1 & 3 & 3 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, or $\underset{\sim}{A}|x\rangle = |\emptyset\rangle$

Test: $|\underset{\sim}{A}| = 1 \cdot (-1)^3 \cdot \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix} + (-1) \cdot (-1)^4 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} + 1 \cdot (-1)^5 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$

$= (-1) \cdot 6 + (-1) \cdot (-3) + (-1) \cdot (-5) = 2.$

$\therefore$ since $|\underset{\sim}{A}| \neq 0$, only solution is trivial: $x = y = z = 0$.

$\boxed{3.2.3}$ Let $\vec{r} = c_1\vec{r}_1 + c_2\vec{r}_2$.

By definition, $\underset{\sim}{A}\vec{r} = \sum_j a_{ij} r_j$

$$= \sum_j a_{ij}(c_1 r_{1j} + c_2 r_{2j})$$

$$= \sum_j (c_1 a_{ij} r_{1j} + c_2 a_{ij} r_{2j})$$ since c_i, a_{ij}, r_j are all numbers

$$= c_1 \sum_j a_{ij} r_{1j} + c_2 \sum_j a_{ij} r_{2j}$$

$$= c_1 \underset{\sim}{A}\vec{r}_1 + c_2 \underset{\sim}{A}\vec{r}_2$$

$\boxed{3.2.20}$ $\underset{\sim}{L}^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\underset{\sim}{L}^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(The vectors $|-1\rangle$, $|0\rangle$, $|+1\rangle$ represent the 3 spin projection states J_z of a spin-1 system $\rightarrow$ $2j+1 = 3$ states.)

$\underset{\sim}{L}^+|-1\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |0\rangle$. (raise by $\Delta m = 1$)

$\underset{\sim}{L}^-|-1\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{\emptyset}$. (lower below m_{min})

$\underset{\sim}{L}^+|0\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle$. (raise by $\Delta m = 1$)

$\underset{\sim}{L}^-|0\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |-1\rangle$. (lower by $\Delta m = -1$)

$\underset{\sim}{L}^+|1\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{\emptyset}$. (raise above m_{max})

$\underset{\sim}{L}^-|1\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |0\rangle$. (lower by $\Delta m = -1$)

where $\vec{\emptyset} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$\boxed{3.2.32}$ Show that if $\underset{\sim}{A}^{-1}$ has elements

$$(\underset{\sim}{A}^{-1})_{ij} = \bar{a}^{1}_{ij} = C_{ji}/|\underset{\sim}{A}| \qquad (|\underset{\sim}{A}| \neq 0)$$

then $\underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{1}$.

We use the definition $\det(\underset{\sim}{A}) = |\underset{\sim}{A}| = \sum_i a_{ij} C_{ij} = \sum_i a_{ji} C_{ji}$.

We look at $(\underset{\sim}{A}\underset{\sim}{B})_{iK} = \sum_j a_{ij} b_{jK}$, where $b_{jK} = C_{Kj}/|\underset{\sim}{B}|$.

For $i = K$, $(\underset{\sim}{A}\underset{\sim}{B})_{KK} = \sum_j a_{Kj} \frac{C_{Kj}}{|\underset{\sim}{A}|} = \frac{1}{|\underset{\sim}{A}|} \sum_j a_{Kj} C_{Kj} = \frac{|\underset{\sim}{A}|}{|\underset{\sim}{A}|} = 1$.

For $i \neq K$, $(\underset{\sim}{A}\underset{\sim}{B})_{iK} = \frac{1}{|\underset{\sim}{A}|}\sum_j a_{ij} C_{Kj}$. As an example, consider

$(\underset{\sim}{A}\underset{\sim}{B})_{12} = \frac{1}{|\underset{\sim}{A}|}\sum_j a_{1j} C_{2j}$ for a 3x3 matrix. The sum represents a determinant with equal 1st and 2nd rows. According to Antisymmetry property one (pg. 169), any det. with equal rows or columns is zero.

In general, $(AB)_{iK} = \frac{1}{|\underset{\sim}{A}|} \sum a_{ij} C_{Kj} = 0$ for $i \neq K$.

$\therefore (\underset{\sim}{A}\underset{\sim}{B})_{iK} = \delta_{iK} \Rightarrow \underset{\sim}{B} = \underset{\sim}{A}^{-1}$ with $\underset{\sim}{B}_{jK} = C_{Kj}/|\underset{\sim}{A}|$

//

$\boxed{3.2.34}$ Find $\underset{\sim}{M}_L \ni M_L \underset{\sim}{A}$ gives $\underset{\sim}{A}$ with

(a) row $i \Rightarrow K \cdot$ row i, K constant.

So $\underset{\sim}{C} = M_L \underset{\sim}{A}$, $\quad c_{jm} = \sum_n m_{jn} a_{nm}$

where $c_{jm} = \begin{cases} a_{jm}, & j \neq i \\ K a_{jm}, & j = i \end{cases}$ so only the i^{th} column is modified from $\underset{\sim}{1}$

$$\therefore \underset{\sim}{M}_L = \begin{array}{c} \begin{array}{ccccc} & & & i & \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & K & 0 \; 0 \\ & & & 0 & 1 \; 0 \\ & & & 0 & 0 \; 1 \end{pmatrix} \end{array}, \text{ or } m_{jn} = \delta_{jn}\left(1 + (K-1)\delta_{in}\right) \quad //$$

verify: $c_{jm} = \sum_n \delta_{jn}(1+(K-1)\delta_{in}) a_{nm} = a_{jm}(1+(K-1)\delta_{ij})$

$$= \begin{cases} a_{jm}, & j \neq i \\ a_{jm}, & j = i \end{cases} \quad ✓$$

(b) row $i \longrightarrow$ row i $-$ $K \cdot$ row m

so $c_{jn} = \begin{cases} a_{jn}, & j \neq i \\ a_{jn} - K a_{mn}, & j = i \end{cases}$

$$\text{so} \quad \underset{\sim}{M}_L = \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & \ddots & \\ & & 1 - \frac{K a_{mn}}{a_{in}} & 0 \\ & & 0 & 1 \; 1 \end{pmatrix} \end{array} \qquad m_{j\ell} = \delta_{j\ell}\left(1 - \left(\frac{K a_{mn}}{a_{\ell n}}\right)\delta_{i\ell}\right)$$

verify: $c_{jn} = \sum_\ell m_{j\ell} a_{\ell n} = \begin{cases} \left(1 - \frac{K a_{mn}}{a_{jn}} \delta_{ij}\right) a_{jn} = a_{jn}, & j \neq i \\ \left(1 - \frac{K a_{mn}}{a_{jn}}\right) a_{jn} = a_{jn} - K a_{mn}, & j = i \end{cases}$ ✓

(c) row $i \longleftrightarrow$ row m $\quad (a_{ij} \longleftrightarrow a_{mj}\ ,\ j = 1, 2, \ldots, n)$

$$C_{jn} = \sum_{\ell} m_{j\ell}\, a_{\ell n}$$

$$\text{Try} \quad m_{j\ell} = \delta_{j\ell}\left[1 + \delta_{i\ell}\cdot\left(\frac{a_{mn}}{a_{\ell n}} - 1\right) + \delta_{m\ell}\cdot\left(\frac{a_{in}}{a_{\ell n}} - 1\right)\right]$$

$$\underset{\sim}{M}_L = \begin{array}{c} \begin{array}{ccccc} & & i & & m \end{array} \\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \frac{a_{mj}}{a_{ij}} & & \\ & & & 1 & \\ 0 & & & & \frac{a_{ij}}{a_{mj}} & \\ & & & & & 1 \end{pmatrix} \end{array}$$

Verify:

$$C_{jn} = \begin{cases} a_{jn}, & j \neq i,\ j \neq m \quad \checkmark \\ \left(1 + \frac{a_{mn}}{a_{in}} - 1\right) a_{in} = a_{mn}, & j = i \\ \left(1 + \frac{a_{in}}{a_{mn}} - 1\right) a_{mn} = a_{in}, & j = m \end{cases}$$

$\boxed{3.2.36}$ $\quad \underset{\sim}{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{row } 1 \to \text{row } 1 - \text{row } 2$$

$\downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 7/2 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \qquad \text{row } 3 \to \text{row } 3 - \tfrac{1}{2}\,\text{row } 2$$

$\downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 7/2 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \qquad \text{row } 2 \to \text{row } 2 - 2\cdot\text{row } 1$$

$\downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7/2 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 \\ -2 & 22/7 & -2/7 \\ 0 & -1/2 & 1 \end{pmatrix} \qquad \text{row } 2 \to \text{row } 2 - \tfrac{2}{7}\,\text{row } 3$$

$\downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 11/7 & -1/7 \\ 0 & -1/7 & 2/7 \end{pmatrix} \qquad \begin{array}{l} \text{row } 2 \to \text{row } 2 \cdot \tfrac{1}{2} \\ \text{row } 3 \to \text{row } 3 \cdot 2/7 \end{array}$$

Check: $\frac{1}{7}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\checkmark$ $\quad \therefore \underset{\sim}{A}^{-1} = \frac{1}{7}\begin{pmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ //

$\boxed{3.3.9}$ general similarity transformation:

$$\underset{\sim}{A}' = \underset{\sim}{B}\,\underset{\sim}{A}\,\underset{\sim}{B}^{-1}$$

Show $Tr(\underset{\sim}{A}') = Tr(\underset{\sim}{A})$.

For any $n \times n$ matrix $\underset{\sim}{M}$, $Tr(\underset{\sim}{M}) \equiv \sum_i m_{ii}$.

$$Tr(\underset{\sim}{A}') = Tr(\underset{\sim}{B}\,\underset{\sim}{A}\,\underset{\sim}{B}^{-1})$$

$$= \sum_i (\underset{\sim}{B}\,\underset{\sim}{A}\,\underset{\sim}{B}^{-1})_{ii}$$

$$= \sum_i \left(\sum_k b_{ik} (\underset{\sim}{A}\,\underset{\sim}{B}^{-1})_{ki} \right)$$ by def. of matrix mult.

$$= \sum_i \left(\sum_k b_{ik} \sum_j a_{kj}\, b^{-1}_{ji} \right)$$ " " " "

$$= \sum_i \sum_k \sum_j a_{kj}\, b^{-1}_{ji}\, b_{ik}$$ since $c_1(c_2+c_3) = c_1c_2 + c_1c_3$ for numbers

$$= \sum_k \sum_j a_{kj} \sum_i b^{-1}_{ji}\, b_{ik}$$

$$= \sum_k \sum_j a_{kj} (\underset{\sim}{B}^{-1}\underset{\sim}{B})_{jk}$$

$$= \sum_k \sum_j a_{kj}\, \delta_{jk}$$

$$= \sum_k a_{kk}$$

$$= Tr(\underset{\sim}{A})$$ //